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On the Distribution of Temperature Relative to Height in Stationary Planetary Waves[1]

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Abstract

It is shown that some gross features of the distribution of the mean temperature relative to the mean height on the normal maps for January can be computed from the vertically integrated linearized vorticity equation, if the effects of surface friction and mountains are included in the equation for the stationary case. By a mathematical analysis supplemented by certain numerical experiments it is shown that it is necessary to assume a meridional scale parameter corresponding to less than 90 degrees of latitude in order to obtain the correct phase-lag. An empirical determination of the meridional wave-length was carried out using 500 mb data for a single month (January 1959).

Section 6 contains some comments on the assumptions made in this study. It is especially pointed out that the effect of horizontal advection of vorticity and the effects of the lower boundary condition (friction and topography) cannot be neglected in comparison with the beta-effect in studies of stationary motion on the planetary scale.

1. Introduction

The main problem in the present paper is the adjustment between the temperature and the height field in planetary waves in the atmosphere. The planetary waves are here understood to be the very long, almost stationary waves observed in the atmosphere on a daily basis and on time-mean maps. The troughs and ridges in these waves are found in certain preferred geographical positions as seen on normal maps. The positions of the waves as studied by a series of daily maps seem to be in the neighborhood of the preferred locations, and one may perhaps best characterize the motion of the waves as oscillations around the mean positions.

The theory for the existence, maintenance and motion of the planetary waves is still fragmentary. CHARNEY and ELIASSEN (1949) have pointed to the importance of the large scale mountains and friction for the understanding of the existence of these waves, while SMAGORINSKY (1953) has investigated the combined effect of large scale heat sources and friction. Recently it has also been suggested that a factor of major importance for the maintenance of the planetary waves against frictional dissipation could be a non-linear transfer of kinetic energy from the smaller scales, where kinetic energy is created by a direct conversion of available potential energy on this (smaller) scale (STARR, 1958, 1959, SALTZMAN, 1959 and WIIN-NIELSEN, 1959).

In the present study we shall assume from the outset that the waves are stationary. We

[1] The main part of the paper was presented at the National Meeting of the American Meteorological Society April 1960, Washington, D.C.

are therefore not dealing with the problem of the motion of the planetary waves. The specific problem to be considered here is the phase and amplitude of the vertically averaged temperature field in the atmosphere relative to the vertically averaged pressure-height in planetary waves. We are therefore dealing with the problem of the thermal structure of the planetary waves. This problem is only a detail of the general problem since it is concerned with the adjustment of temperature relative to height in already existing waves and therefore assumes that the waves are maintained by some process.

2. Representation of Temperature and Height

It was found preferable to restrict the main part of the investigation to the mean temperature and mean height. The two fields were found following the procedure originally outlined by ELIASSEN (1956). The height field is represented by the following expression

$$Z(x,y,p,t) = \bar{Z}(x,y,t) + A(p) Z_T(x,y,t) \quad (2.1)$$

The function $A(p)$ was determined by the formula given by Eliassen:

$$A(p) = \frac{U - \bar{U}}{\left\{\overline{(U-\bar{U})^2}\right\}^{1/2}} \quad (2.2)$$

$U(p)$ is a representative zonal wind profile. In this case $U(p)$ was taken from Buch's data for winter (1954). The bar over a quantity means the average with respect to pressure, i.e.,

$$\overline{(\quad)} = \frac{1}{p_g}\int_0^{p_g} (\quad) dp \quad (2.3)$$

where p_g is the surface pressure ($p_g = 100$ cb).

The wind $U = U(p)$ and the function $A(p)$ determined from (2.2) are given in Table 1.

It is seen by direct evaluation that the function $A(p)$ defined by (2.2) satisfy the requirements discussed by ELIASSEN (1956) that

$$\bar{A} = 0, \ \overline{A^2} = 1 \quad (2.4)$$

The mean height field can be determined directly from the definition of the bar operator if we know the height field for sufficiently many isobaric surfaces. In our case the normal maps for the 1,000, 850, 700, 500, 300, 200,

Table 1

p	$U(p)$, m sec^{-1}	$A(p)$
0	(0)	−1.8
10	15.5	0.7
20	19.1	1.3
30	18.0	1.1
40	15.8	0.8
50	13.6	0.4
60	10.8	−0.03
70	8.1	−0.5
80	5.7	−0.9
90	3.3	−1.3
100	2.1	−1.7

$\bar{U} = 11.0$ m sec^{-1}
$U_T = 6.1$ m sec^{-1}

and 100 mb prepared by JACOBS (1958), WEGE (1957), and the U.S. WEATHER BUREAU (1952) were used to define the mean height field along 50° N in January. The deviation from the mean height along 50° N is represented in figure 1 as the solid curve.

If we multiply (2.1) by $A(p)$ and next apply the bar operator it is seen that

$$Z_T(x,y,t) = \frac{1}{p_g}\int_0^{p_g} A(p) Z dp \quad (2.5)$$

Using the same height fields as before and the function $A(p)$ given in Table 1, we can compute Z_T. The deviation from the average value along 50° N is given in figure 1 as the dashed curve.

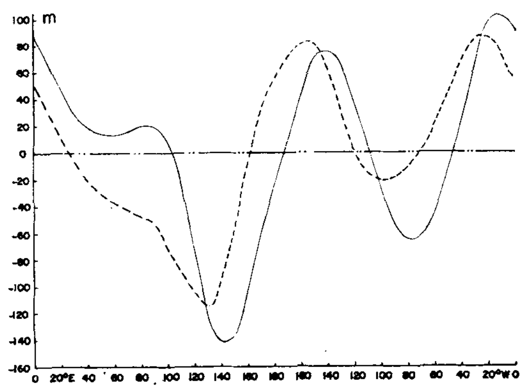

Fig. 1. The solid curve is the deviation from the latitudinal average of the mean height field, $\bar{Z}$, as a function of longitude for 50° N, prepared from normal data for January. The dashed curve is the corresponding curve for the mean temperature, Z_T.

The mean height field $\overline{Z}$ shows the positions of the two major troughs along about 140° E and 80° W and the minor trough along 50° E. The thermal field Z_T shows to a very large extent the same configuration, but with the distinct difference that the thermal trough is lagging behind the height trough by an amount which varies from about 10° of longitude in the trough along the eastern coast of Asia to 20—25° of longitude in the trough along the east coast of North America.

If we knew the details of the laws for the heating of the atmosphere, which would have to be specified for all the different components of heating (radiation, heat of condensation and evaporation, heat exchange with the underlying surface, etc.) we could try to investigate whether it was possible to compute the two curves in figure 1 from a knowledge of the heating, the mountain effect, and the frictional law. Lacking this knowledge we may try to see how much we can say about the relative distribution of temperature and height considering only the dynamical equations and disregard completely the thermodynamics of the atmosphere. We shall treat this problem in the next section.

3. The Dynamical Equation

The dynamical model to be used in the following will be essentially the one given by Phillips (1958), but with the important modification that we shall incorporate the influence of the lower boundary condition and in this way pay attention to the effects of the topographical features of the earth and of friction. As we are not here going to consider the effects of the heating we may restrict the investigation to the vertically integrated vorticity equation. We shall use the vorticity equation in its simplest form:

$$\frac{\partial \zeta}{\partial t} + \mathbf{v} \cdot \nabla (\zeta + f) = f_0 \frac{\partial \omega}{\partial p} \qquad (3.1)$$

and shall further assume that the horizontal wind and the vorticity may be computed from the geostrophic assumption. The symbols in (3.1) have the following meaning: $\mathbf{v}$ is the horizontal wind vector, ζ the relative vorticity, f the Coriolis' parameter (f_0 a standard value) and $\omega = dp/dt$ the vertical velocity.

Introducing the expression (2.1) in (3.1) and averaging over the complete depth of the atmosphere we get:

$$\frac{\partial \bar{\zeta}}{\partial t} + \bar{\mathbf{v}} \cdot \nabla(\bar{\zeta} + f) + \mathbf{v}_T \cdot \nabla \zeta_T = \frac{f_0}{p_g} \omega_g \qquad (3.2)$$

where ω_g is the value of (dp/dt) at the ground.

ω_g may now be related to the topography of the earth and the friction in the following way:

$$\omega_g = \left(\frac{\partial p}{\partial t}\right)_g + \mathbf{v}_g \cdot \nabla p_g - W_g \varrho_g g \qquad (3.3)$$

In our geostrophic model the pressure advection, $\mathbf{v}_g \cdot \nabla p_g$ will vanish. Hydrostatically, we may write

$$\left(\frac{\partial p}{\partial t}\right)_g = g \varrho_g \left(\frac{\partial z}{\partial t}\right)_g \qquad (3.4)$$

which introduced in (3.3) results in

$$\omega_g = g \varrho_g \left[\left(\frac{\partial Z}{\partial t}\right)_g - W_g\right] \qquad (3.5)$$

The vertical velocity, W_g, at the ground may now be considered as composed of two parts:

$$W_g = W_t + W_f \qquad (3.6)$$

where W_t is the vertical velocity due to the sloping terrain, while W_f is the vertical velocity produced by the skin friction. The vertical velocity due to the mountains will be computed from the following expression, which assumes that the height of the mountains is small:

$$W_t = \mathbf{v}_g \cdot \nabla h \qquad (3.7)$$

where h is the height of the ground over mean sea level and $\mathbf{v}_g$ the horizontal wind at the ground, while the second component, W_f, will be computed following the procedure by Charney and Eliassen (1949), which results in the following expression:

$$W_f = F^{\star} \zeta_g \qquad (3.8)$$

where $F^{\star}$ is assumed to be a constant.

(3.5) may now be written:

$$\omega_g = g \varrho_g \left[\left(\frac{\partial z}{\partial t}\right)_g - \mathbf{v}_g \cdot \nabla h - F^{\star} \zeta_g\right] \qquad (3.9)$$

As we are concerned here with certain features of the long-term average of the atmospheric height and temperature fields we may assume that the flow is stationary. When this assumption is introduced, and when we further substitute from (3.9) into (3.2) we obtain the following dynamic equation:

$$\bar{\mathbf{v}} \cdot \nabla(\bar{\zeta} + f) + \mathbf{v}_T \cdot \nabla \zeta_T = = -\frac{g f_0}{R T_g}(\mathbf{v}_g \cdot \nabla h + F^{\star} \zeta_g) \qquad (3.10)$$

According to our model approximations we may obtain the quantities at the ground (assumed to be closed to 100 cb) from (2.1), i.e.,

$$\begin{cases} \mathbf{v}_g = \bar{\mathbf{v}} + A_g \mathbf{v}_T \\ \zeta_g = \bar{\zeta} + A_g \zeta_T \end{cases} \qquad (3.11)$$

where A_g is the value of $A(p)$ for $p = p_g =$ 100 cb.

Due to (3.11) and our geostrophic assumption we may consider (3.10) as an equation with two unknowns, $\bar{Z}$ and Z_T. Another equation relating $\bar{Z}$ to Z_T could be derived from a combination of the thermal vorticity equation and the thermodynamic equation, but this equation would involve the heating and is not considered here, according to the discussion in section 2.

In order to study the relative phase and amplitude of the temperature and height field we may, however, proceed in the following way. One of these two fields may be assumed to be known, say $\bar{Z}$. From the dynamical equation we may then find the solution for Z_T. The resulting field of Z_T can then be compared with the observed distribution of Z_T. (3.10) is extremely laborious to solve in its non-linear form. Due to this fact we may proceed to linearize the equation and obtain solutions to this much simpler equation.

When the linearization has been made, the geostrophic assumption introduced in the resulting equation, and we further have made use of (3.11) we may write the equation in the form:

$$\bar{U}\frac{\partial \nabla^2 \bar{Z}}{\partial x} + \beta \frac{\partial \bar{Z}}{\partial x} + U_T \frac{\partial \nabla^2 Z_T}{\partial x} = = -\frac{f_0^2}{R T_g} U_g \frac{\partial h}{\partial x} - F(\nabla^2 \bar{Z} + A_g \nabla^2 Z_T) \qquad (3.12)$$

Due to the fact that atmospheric disturbances have a limited meridional scale we shall introduce a meridional scale factor m in the evaluation of the Laplacian. We may thus write the Laplacian operator in the form:

$$\nabla^2 = \frac{d^2}{dx^2} - m^2 \qquad (3.13)$$

(CHARNEY and ELIASSEN, 1949).

Introducing (3.13) in (3.12) we can write the final equation in the form:

$$\frac{d^3 Z_T}{dx^3} + \frac{F A_g}{U_T}\frac{d^2 Z_T}{dx^2} - m^2 \frac{d Z_T}{dx} - \frac{F A_g m^2}{U_T} Z_T = H(x) \qquad (3.14)$$

where $H(x)$ is given by the expression:

$$H(x) = -\left[\frac{\bar{U}}{U_T}\frac{d^3 \bar{Z}}{dx^3} + \frac{F}{U_T}\frac{d^2 \bar{Z}}{dx^2} + \frac{\beta \bar{U} - m^2}{U_T}\frac{d\bar{Z}}{dx} - \frac{F m^2}{U_T}\bar{Z} + \frac{f_0^2}{R T_g}\frac{U_g}{U_T}\frac{dh}{dx}\right] \qquad (3.15)$$

The equation (3.14) with the expression (3.15) for the function $H(x)$ represents the simplest possible system which incorporates the effects of mountain and friction. When we apply equation (3.14) to the real atmosphere we have disregarded the non-linear effects, and we have further to determine the value to be used for the frictional coefficient F and the meridional scale factor m.

With respect to the frictional coefficient we have some uncertainty regarding the numerical value. SMAGORINSKY (1953) used a value for F of 2×10^{-6} sec^{-1}. CHARNEY and ELIASSEN (1949) made their investigation for two different values, $F = 2 \times 10^{-6}$ sec^{-1} and $F = 4 \times 10^{-6}$ sec^{-1}. The latter value was adopted by PHILLIPS (1956) in his investigation of the general circulation. The values for F quoted so far were obtained by an application of Ekman's theory for friction in the surface layer. MINTZ (1956) has tried to determine the value of F using the vertically integrated vorticity equation. He finds first of all a variation of F over the map. More important perhaps is the fact that he finds a mean value of F, which is about four times larger than the one adopted by Phillips or about 16×10^{-6} sec^{-1}. It appears therefore difficult to determine a value of F to be used in our computations, because the values suggested so far vary by almost one order of magnitude. In view of this it was decided to make the computation for several values of F and determine the sensitivity of the computation to these values.

CHARNEY and ELIASSEN (1949) as well as SMAGORINSKY (1953) used a scale factor, m, defined as in this investigation. The values which these investigators adopted varied between a meridional wave-length of 66 and 30 degrees of latitude based on the lateral width of the Rocky Mountains and the major Fourier components of the meridional variation of the heat sources and sinks.

A characteristic value of m may be found from atmospheric data in the following way. The representation (3.13) assumes that the fields $\bar{Z}$ and Z_T may be written in the following form:

$$Z = \cos(my)g(x) \tag{3.16}$$

from which it follows that

$$\nabla^2 Z = \frac{\partial^2 Z}{\partial x^2} - m^2 Z \tag{3.17}$$

m^2 can therefore be determined empirically from the formula

$$m^2 = -\frac{1}{Z}\frac{\partial^2 Z}{\partial y^2} \tag{3.18}$$

Characteristic values of m were determined by evaluating the right hand side of (3.18) along latitude circles and next obtaining mean values of m by averaging the individual values. The computation was performed for 500 mb maps for each day of the month of January 1959. The meridional wave-length corresponding to m $(m = 2\pi/D)$ is given in Table 2 as a function of latitude. The values in Table 2 are obtained by averaging the results from the individual days over the month.

Table 2

Latitude	30°	40°	50°	60°	70°
Wave-length, km	6,500	2,900	4,300	5,500	8,700

It is seen in Table 2 that the effective meridional wave-length in the middle latitudes varies from about 3,000 to 6,000 km, which agrees well with the values used by CHARNEY and ELIASSEN (1949) and SMAGORINSKY (1953). The values of m used in the present calculations were varied in order to find the sensitivity of the computations to the scale factor (see Section 5).

4. The Adjustment of Temperature to Height in Sinusoidal Waves

In this section we are going to disregard the mountain effect in order to gain some insight into the phase and amplitude of the temperature wave relative to the height wave in simple sinusoidal waves. We are in other words going to assume that:

$$\bar{Z} = \bar{a} \sin kx \tag{4.1}$$

and

$$Z_T = a_T \sin(kx + \alpha_T) \tag{4.2}$$

and we want to find $a_T/\bar{a}$, the relative amplitude, and α_T the relative phase.

Substituting (4.1) and (4.2) in (3.14) and (3.15) we find that the following set of linear equations have to be satisfied:

$$U_T[a_T \cos\alpha_T] + \frac{FA_g}{k}[a_T \sin\alpha_T] = \left(\frac{\beta}{m^2+k^2} - \bar{U}\right)\bar{a} \tag{4.3}$$

$$-\frac{FA_g}{k}[a_T \cos\alpha_T] + U_T[a_T \sin\alpha_T] = \frac{F}{k}\bar{a} \tag{4.4}$$

The solutions to (4.3) and (4.4) may be written in the form:

$$\frac{a_T}{\bar{a}} = \left(\frac{C^2_R + F^2/k^2}{U_T^2 + A_g^2 F^2/k^2}\right)^{\frac{1}{2}} \tag{4.5}$$

$$\tan\alpha_T = -\frac{F}{k}\cdot\frac{U_T - A_g C_R}{U_T C_R + A_g (F/k)^2} \tag{4.6}$$

In (4.5) and (4.6) we have introduced the notation:

$$C_R = \bar{U} - \frac{\beta}{m^2+k^2} \tag{4.7}$$

The relations (4.5) and (4.6) can be used to study the relative amplitude and relative phase as a function of the frictional coefficient F and the zonal and meridional scale.

It will be noted that in the absence of friction we find $\tan\alpha_T = 0$ which means that the thermal waves is either in phase or 180° out of phase with the height wave. In the same case we find that

$$\frac{a_T}{\bar{a}} = \frac{|C_R|}{U_T} \tag{4.8}$$

In the evaluation of (4.5) and (4.6) as a function of wave-numbers and the frictional coefficient the following parameters were defined:

$$N = ak \cos \varphi \tag{4.9}$$

$$M = \frac{a}{2} m \tag{4.10}$$

where a is the radius of the earth. N is the number of waves in the zonal direction around the hemisphere, while M is the number of waves in the meridional direction in a hemisphere, i.e., from equator over the pole to equator. Figure 2a—d shows isolines for $a_T/\bar{a}$ as a function of M and N. Figure 2a corresponds to the case of no friction, while figure 2b—d corresponds to increasing values of F. Figure 3a—d shows in a similar way the phase difference α_T as a function of the wave-numbers M and N for increasing values of the frictional coefficient F. The curves in figures 2 and 3 were computed with the values for $\bar{U}$, U_T, U_g, and A_g given in Table 1.

Several interesting facts regarding the importance of friction for the adjustment of temperature and height in planetary stationary waves can be seen from figures 2 and 3. With respect to the amplitude ratio $a_T/\bar{a}$ we find

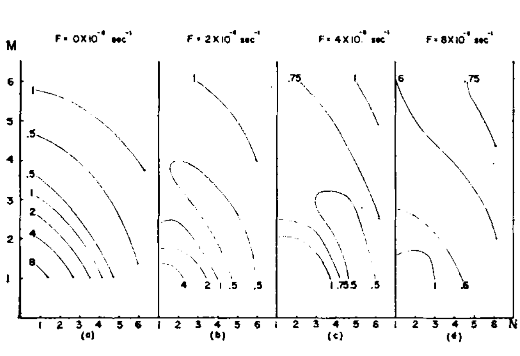

Fig. 2. Isolines of the amplitude ratio, $a_T/\bar{a}$in, sinusoidal stationary waves for different intensities of friction. The horizontal coordinate is the number of waves around the latitude circle 50° N, while the vertical coordinate is the number of waves in the hemisphere in the meridional direction.

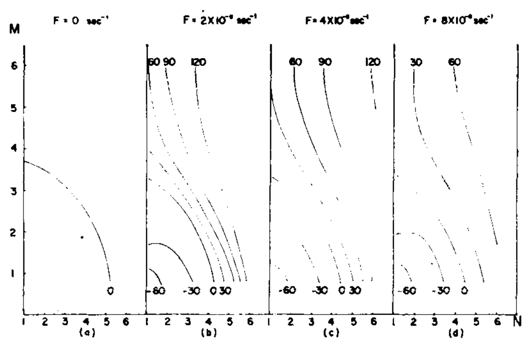

Fig. 3. Isolines of the phase difference between the thermal wave and the wave in the mean height field in sinusoidal, stationary waves for different intensities of friction. Coordinates as in figure 2.

that this number has its largest value for waves, which are very long in the meridional direction as well as in the zonal direction. When the wave-lengths in the two directions decrease, the amplitude ratio first decreases to a minimum, but for waves which have a rather short wave-length in both directions, the ratio starts to increase again. We notice further that the amplitude ratio is sensitive to the magnitude of the frictional coefficient, especially for waves which are very long in both directions. In general the amplitude ratio decreases as the friction increases. It is also evident from figure 2 that there is no possibility to explain the observed distribution of Z_T relative to $\bar{Z}$ from a frictionless theory. The amplitude ratio would be too large.

If, on the other hand, the frictional coefficient was very large ($F \to \infty$) we would again have tan $\alpha_T = 0$ and $a_T/\bar{a} = 1/|A_g| \simeq 0.6$ indicating that the thermal wave would be in phase with the mean height wave, but with an amplitude much smaller than observed in reality.

Figure 3a shows the dividing line between the region where the thermal field would be in phase, and the region where the thermal field would be 180° out of phase with the height field. As the friction increases we notice that the dividing line ($\alpha_T = 0$) stays approximately in the same location in the diagram. On the long wave side of the dividing lines we find $\alpha_T < 0$, which means that the temperature field will be ahead of the height field in waves which are long in the meridional direction as well as the zonal, while $\alpha_T > 0$ on the short wave side of the dividing line. Here, the temperature field will be lagging behind the height field. For a small value of the friction we find a large lag for waves which are short in the meridional direction (M large), but this lag decreases as the friction is increased.

From the diagrams in figures 2 and 3 it is seen that it is rather unlikely that we can explain the observed distribution of temperature relative to height, if the meridional scale is very large, corresponding to $M=1$ to 2, and the frictional coefficient F is small. In this case (see figures 2b and 3b) we would obtain an amplitude ratio, which is somewhat larger than 1, and we would further find the temperature would precede the height field for the large scale component in the zonal direction. Both of these statements are not in agreement with the observed distribution as shown in figure 1.

The observed distribution of temperature relative to height can therefore only be explained from this linear theory if the meridional scale can be assumed to correspond to $M \geq 3$. $M=3$ corresponds to a meridional wave-length of about 60° of latitude. This meridional scale is certainly a major one in the flow close to the surface, and an inspection of BUCH's (1954) analysis of the mean meridional wind averaged in time shows that scales of this order of magnitude also are present on the higher levels.

In the next section we shall investigate the results of a computation of the temperature distribution (the dashed curve of figure 1) from the mean height distribution (the solid curve of figure 1).

5. Computation of the Temperature Distribution in Winter

The equation (3.14) with the expression (3.15) for the right hand side can be solved for Z_T if we know $\bar{Z}$ and h. In this case $\bar{Z}$ was taken from the curve on figure 1 giving the height variation along 50° N for the January normal. The mountain height h was obtained from the maps prepared by BERKOFSKY and BERTONI (1955). Equations (3.14) and (3.15) were put into finite differences using the ordinary approximations with a grid length corresponding to 10 degrees of longitude, which gives us 36 grid points. In this formulation, (3.14) is equivalent to a system of 36 linear equations, which may be solved by a matrix calculation. This technique was used in the solution of the problem and was preferred over the relaxation technique due to the rather few grid points.

According to the earlier discussion, the meridional scale was set equal to the equivalent of 45 degrees of latitude ($M=4$) in order to see whether it was possible to account for the main pattern of the thermal field by a solution of (3.14). The computation was made for three different values of the frictional coefficient $F=0$, $F=2\times10^{-6}$ sec^{-1}, and $F=4\times10^{-6}$ sec^{-1}. The results of these computations are shown in figures 4, 5, and 6. The frictionless case (figure 4) indicates clearly what was already expected from section 4, namely that the thermal field is essentially out of phase with the mean height field and with an amplitude, which is too large compared to the observed distribution of Z_T. Comparing next figure 4 and figure 5 we find that the introduction of a moderate frictional coefficient has the effect of decreasing the amplitude of the thermal field and further to decrease the phase lag. However, the computed Z_T field is still displaced too much toward the west compared to the observed distribution of Z_T.

When we increase the frictional coefficient to 4×10^{-6} sec^{-1} (figure 6) we find no further

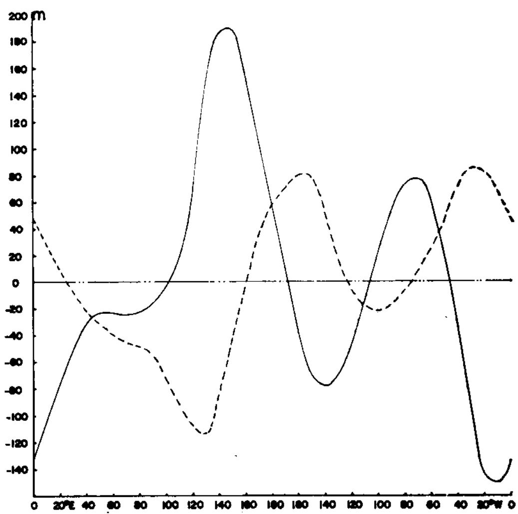

Fig. 4. The dashed curve is the observed distribution of the mean temperature, Z_T, reproduced from figure 1. The solid curve is computed from eq. (3.14). Parameters: $\bar{U}=11$ m sec^{-1}, $U_T=6.1$ m sec^{-1}, $M=4$, $F=0\times10^{-6}$, sec^{-1}.

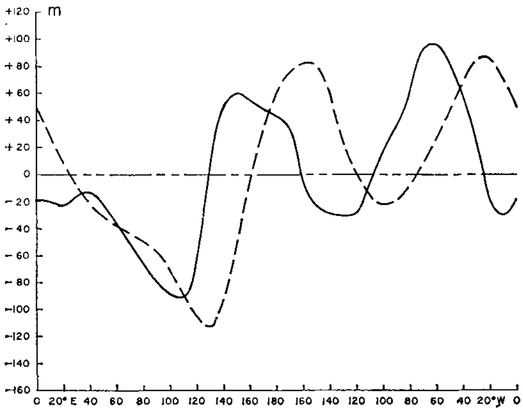

Fig. 5. Same as fig. 4. Parameters: $\overline{U}=11$ m sec^{-1}, $U_T=$ $=6.1$ m sec^{-1}, $M=4$, $F=2\times10^{-6}$ sec^{-1}.

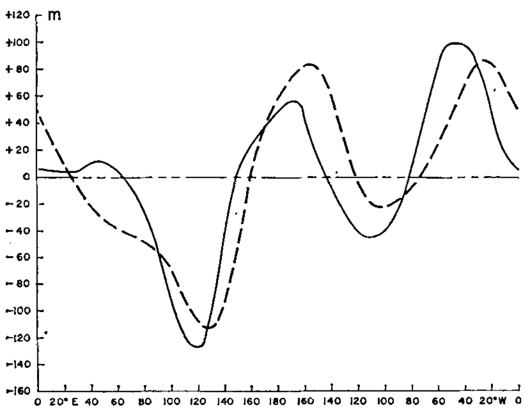

Fig. 6. Same as fig. 4. Parameters: $\overline{U}=11$ m sec^{-1}, $U_T=6.1$ m sec^{-1}, $M=4$, $F=4\times10^{-6}$ sec^{-1}.

change in the amplitude, but the increased value of F has the effect of decreasing the phase lag between the computed Z_T field and the mean height field $\overline{Z}$. A comparison between the computed and observed field of Z_T now shows an agreement which is about as good as we can expect considering the very simple linear theory applied in these computations.

A number of other experiments using larger values of the meridional scale was made. They result in computed Z_T-fields, which by and large agree with the results obtained in the preceding section. Figure 7 shows a case where the meridional scale is 90 degrees of latitude ($M=2$) and $F=4\times10^{-6}$ sec^{-1}. The main difference between the computed and observed Z_T-field is that the larger meridional scale now forces the computed Z_T-field to have its troughs and ridges displaced somewhat toward the east compared to the observed field. This is especially noticeable in the ridge along 150° W and the trough around 100° W.

6. Comments on the Assumptions

The treatment given of the present problem in the preceding sections is admittedly extremely simplified. The two-parameter model used in the calculations is not in itself a restriction because the problem has been formulated within the framework of the model since we have restricted ourselves to a consideration of the mean thickness and mean height field. More severe is probably the linearization which has been made of the dynamic equation (3.10). The equation was linearized because it so far has not been possible to solve the complete, non-linear, two-dimensional equation. It is, however, likely that a smaller degree of linearization still allows a solution of the problem. A later investigation of the possibilities for less restrictive assumptions regarding the zonal winds as functions of latitude and the meridional scale of the disturbances may give a further insight into the adjustment between the mean temperature field and the mean height field.

One of the important factors in the present treatment is the surface friction, which is taken proportional to the surface vorticity. Even if this representation may be sufficiently accurate for the problem, it is a further assump-

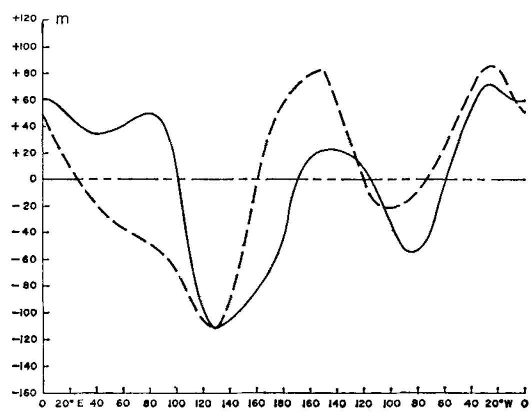

Fig. 7. Same as fig. 4. Parameters: $\overline{U}=11$ m sec^{-1}, $U_T=$ 6.1 m sec^{-1}, $M=2$, $F=4\times10^{-6}$ sec^{-1}.

tion that the surface-wind and vorticity can be expressed with sufficient accuracy by a linear combination of the mean height and mean temperature field using equation (3.11). We may test the validity of this assumption by computing the height field of the 1,000 mb surface using the expression

$$Z_g = \overline{Z} + A_g Z_T \qquad (6.1)$$

and comparing the computed profile (6.1) with the observed profile, Z_{obs}, of the 1,000 mb surface. The two curves are shown in figure 8. It is seen that they compare fairly well although there is a tendency to get a slightly larger amplitude in the computed field. We may therefore state that the surface height field can be expressed quite accurately using two parameters.

The effects of surface friction was taken into account by CHARNEY and ELIASSEN (1949) using the assumption that the surface flow in agreement with the equivalent barotropic assumption was parallel to the 500 mb flow and a certain fraction (0.4) of this flow. A comparison of figure 1 and figure 8 shows that the flow patterns at the 1,000 mb level are systematically displaced toward the east relative to the vertical mean flow, $\overline{Z}$, corresponding to a westward tilt of the systems. It appears therefore that the results obtained by Charney and Eliassen may be partly fortuitous.

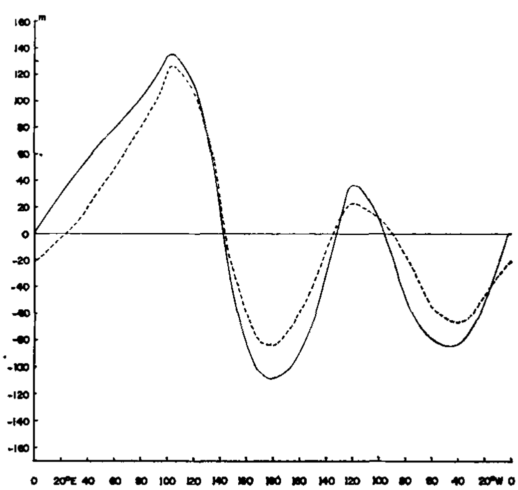

Fig. 8. The observed profile of the 1,000 mb surface along 50° N for the normal map of January (solid curve) compared to the 1,000 mb profile computed from the two-parameter assumption (dashed curve).

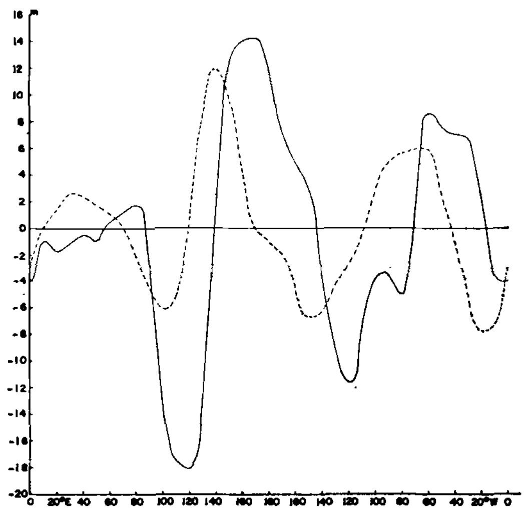

Fig. 9. The relative vorticity, measured by the second finite difference, of the 1,000 mb surface for the normal map of January computed from the two-parameter assumption (eq. (6.1)) compared to the relative vorticity of the mean height field multiplied by 0.4.

The second finite difference in the zonal direction (a measure of relative vorticity) of the 1,000 mb surface computed from the two-parameter model (the solid curve in figure 8) is reproduced in figure 9, as the solid curve. The dashed curve in the same figure is the corresponding measure of relative vorticity computed from the $\overline{Z}$-field and multiplied by 0.4. The displacement towards the west of the latter curve relative to the former is evident.

BURGER (1958) has recently made a scale analysis of the vorticity equation for the planetary flow in a frictionless and adiabatic atmosphere. One of his conclusions is that the vorticity equation for the planetary scale reduces to

$$\beta v + f \nabla \cdot \mathbf{v} = 0 \qquad (6.2)$$

If we were to apply Burger's argument to the vertical mean flow with $\omega = 0$ for $p = p_g$ the vertically integrated vorticity equation would reduce to

$$\bar{v} = 0 \qquad (6.3)$$

while the thermal vorticity equation in the two-parameter case would be

$$\beta v_T + f \nabla \cdot \mathbf{v}_T = 0 \qquad (6.4)$$

and the adiabatic equation

$$\bar{u}\frac{\partial \phi_T}{\partial x} + \bar{v}\frac{\partial \phi_T}{\partial y} + \overline{\sigma B^2}\,\nabla \cdot \bar{\mathbf{v}}_T = 0 \qquad (6.5)$$

$\sigma = -\alpha \partial \ln \Theta / \partial p$ is a measure of static stability and $B(p) = \int_0^p A(p)dp$ where $A(p)$ is defined by (2.1). Eliminating $\nabla \cdot \mathbf{v}_T$ from (6.4) and (6.5) and linearizing we obtain

$$\left(\bar{U} - \overline{\sigma B^2}\frac{\beta}{f^2}\right)\frac{\partial \phi_T}{\partial x} = \Gamma \bar{v} = 0 \qquad (6.6)$$

where $\Gamma = -\frac{\partial \phi_T}{\partial y}$ is a measure of the meridional temperature gradient. From (6.6) it follows in the quasi-geostrophic case that

$$v_T = 0 \qquad (6.7)$$

and no thermal flow could exist.

The next natural step from Burger's analysis will be to incorporate the vertical velocity at the lower boundary caused by friction and mountains, but still neglect the advections of the relative vorticities. The vertically integrated vorticity equation may in this case be obtained from (3.14) and (3.15) by neglecting all terms except those related to the beta-effect, friction and mountains. We obtain:

$$\frac{d^2 Z_T}{dx^2} - m^2 Z_T = -\frac{1}{A_g}\left[\frac{d^2 \bar{Z}}{dx^2} - m^2 \bar{Z} + \frac{\beta}{F}\frac{d\bar{Z}}{dx} + \frac{f^2}{RT_g}\frac{U_g}{F}\frac{dh}{dx}\right] \qquad (6.8)$$

Taking again the simple case in which we neglect the effect of the mountains, but retain the frictional effects we obtain substituting (4.1) and (4.2) into (6.8):

$$\tan \alpha_T = -\frac{\beta}{k^2 + m^2}\frac{k}{F} \qquad (6.9)$$

$$\frac{a_T}{a} = \frac{1}{|A_g|}\left[1 + \left(\frac{\beta}{k^2 + m^2}\right)^2 \frac{1}{(F/k)^2}\right]^{\frac{1}{2}} \qquad (6.10)$$

From (6.9) it is seen that α_T always will be negative indicating that we will have the temperature field preceding the height field, which is contrary to the observed distribution in winter. It follows therefore that the advection of relative vorticity plays an important role in the adjustment between the temperature field and height field.

In view of the conclusions obtained theoretically above it becomes interesting to compare the magnitude and distribution of the different terms in the vorticity equation to the field of vertical velocity (or divergence) for a large scale flow. The normal maps for January were selected as representing a possible stationary flow containing only the planetary scales.

The vertical velocity may be computed from the vorticity equation in the stationary case by integrating from the lower boundary and upwards. We get

$$\omega = \frac{\beta}{f}\int_{p_g}^{p} v\,dp + \frac{1}{f}\int_{p_g}^{p} \mathbf{v} \cdot \nabla \zeta\,dp + \omega_g \qquad (6.11)$$

where ω_g is the vertical velocity at the ground. We may write

$$\omega_g = -g\varrho_g(W_f + W_t) \qquad (6.12)$$

where

$$W_f = \frac{H}{f}F\zeta_g \qquad (6.13)$$

and

$$W_t = \frac{g}{f}J(Z_g, h) \qquad (6.14)$$

W_f and W_t are the vertical velocities due to friction and mountains, respectively, H is the height of the homogeneous atmosphere, Z_g the height of the 1,000 mb surface, and h is the height of the mountains.

Formally, we can write (6.11) in the form

$$\omega = \omega_\beta + \omega_a + \omega_f + \omega_t \qquad (6.15)$$

$$\left\{\begin{aligned} \omega_\beta &= \frac{g\beta}{f^2}\frac{\partial}{\partial x}\left[\int_{p_g}^{p} Z\,dp\right] \\ \omega_a &= \frac{g}{f^2}\int_{p_g}^{p}\{J(Z, \zeta)\}\,dp \\ \omega_f &= -g\,\varrho_g\frac{H}{f}F\zeta_g \\ \omega_t &= -g\varrho_g\frac{g}{f}J(Z_g, h) \end{aligned}\right. \qquad (6.16)$$

We may now measure the contribution from the different terms by computing their contribution to the total vertical velocity.

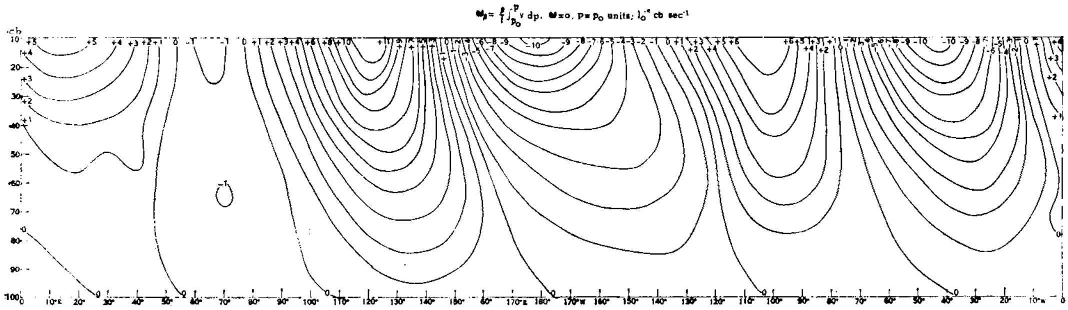

Fig. 10. A zonal cross-section along 50° N of the vertical velocity, ω_β. January normal maps.

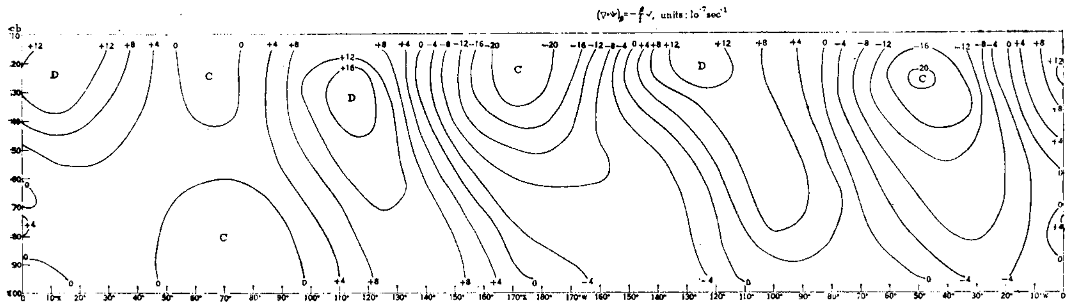

Fig. 11. The divergence in the zonal cross-section computed from the distribution in fig. 10.

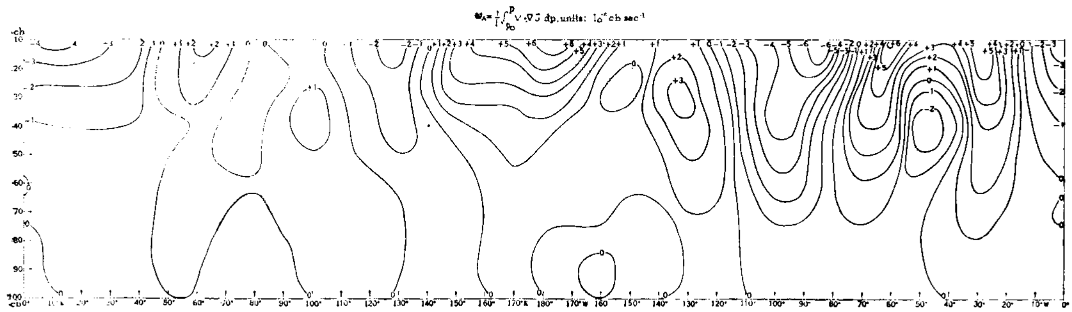

Fig. 12. A zonal cross-section along 50° N of the vertical velocity, ω_a. January normal maps.

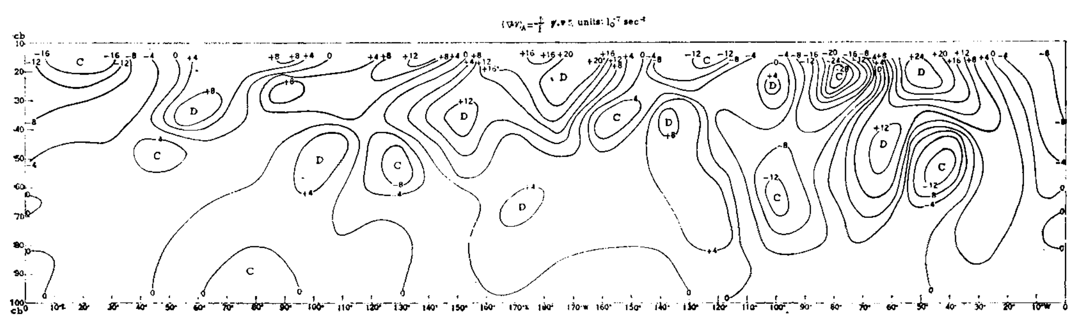

Fig. 13. The divergence in the zonal cross section computed from the distribution in fig. 12.

Figure 10 shows a zonal cross section along 50° N of ω_β, while figure 11 is the divergence computed from the distribution given in fig. 10. These two figures should be compared with figures 12 and 13 which show the distribution of ω_a and the corresponding divergence. It is evident that ω_a is not negligible compared to ω_β and further that the two terms have a tendency to counteract each other. It therefore turns out that the vorticity advection gives a significant contribution to the total vertical velocity and divergence.

The vertical velocity due to friction and mountains enters only at the lower boundary in our formulation. Figures 14 and 15 show the vertical velocity due to these effects as a function of longitude at 50° N. Notice in particular that the vertical velocities due to friction are generally larger than those due to the mountains. The frictional vertical velocity over the Pacific Ocean is comparable with ω_a in order of magnitude.

Figures 16 and 17 show the distribution of vertical velocity and divergence including all four terms in (6.15). Comparing these figures with figures 10 and 11, respectively, it is seen that ω_β is greatly modified by the three other terms.

7. General Conclusions

It has been shown in the preceding sections that some gross features of the distribution of the mean thermal field relative to the mean height field can be described by a linearized form of the vertically integrated vorticity equation. An estimate of a typical meridional scale for mid-tropospheric disturbances obtained from 500 mb data for January 1959 shows that the scale in the average for middle latitudes varies between 3,000 and 6,000 km. A computation of the field of vertical motion in a zonal cross section along 50° N based upon the normal maps for January shows that the effects of horizontal vorticity advection, friction, and mountains cannot be neglected compared with the β-effect even for the planetary scale.

8. Acknowledgements

The present investigation was initiated in cooperation with Professor Norman Phillips, Massachusetts Institute of Technology. The author wants to thank Dr. Phillips for his valuable contributions to the formulation of the problem, for many discussions of the partial results and his kind criticism of an

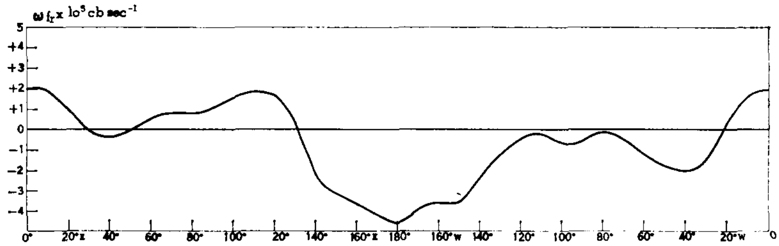

Fig. 14. The vertical velocity due to friction ω_f as a function of longitude at 50° N. January normal maps.

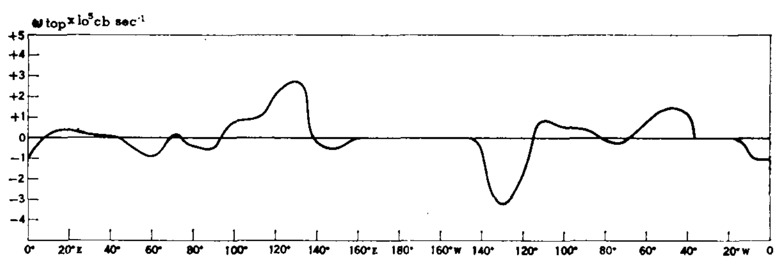

Fig. 15. The vertical velocity due to mountains as a function of longitude at 50° N. January normal maps.

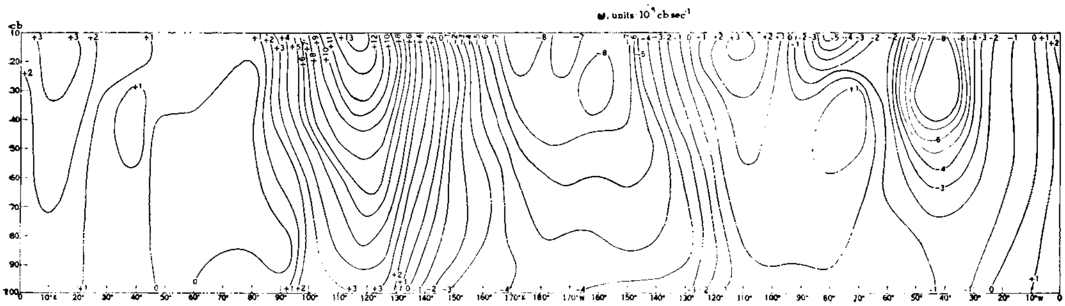

Fig. 16. Distribution of vertical velocity computed from eq. (6.15) in a zonal cross section at 50° N.

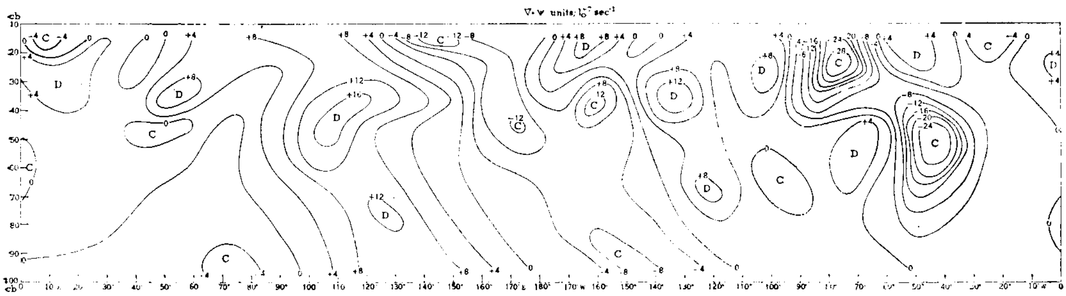

Fig. 17. Distribution of divergence computed from the vertical velocity in fig. 16.

earlier form of this paper. Captain John A. Brown, Jr., kindly coded the computations necessary for the evaluation of the vertical velocities described in section 6, while Mrs. Lena Loman took care of the coding of the other computations in this paper.

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