Tropospheric Geopotential and Velocity Variances over the Arctic

By WARREN L. GODSON, Meteorological Service of Canada, Toronto,

and

M. AYLMER MacFARLANE, Arctic Meteorology Research Group, McGill University, Montreal

(Manuscript received January 11, revised version May 20, 1960)

Abstract

As an elaboration of recent studies by BUELL (1954, 1957) and JENKINSON (1956), the relation between the standard deviation of contour height and the vector standard deviation of geostrophic wind velocity is formulated, both for point values (variations in time) and for chart values (variations in space). The relation between wind and contour variances for a simple schematic contour pattern is examined, from the point of view of both space and time statistics.

The relationship between the space variance of contour height (S^2) and the space variance of the geostrophic wind (σ^2) was evaluated over the Arctic (north of about 60° N) at the 500-mb level. A high degree of correlation was found to exist between these two quantities, although the form of the relationship (σ^2 proportional to S) did not agree with the results of previous studies (BUELL, JENKINSON, loc.cit.) of time variances. However, a reanalysis of such data for northern latitudes yielded results virtually identical with those of the present study.

In both cases, the standard error of estimate was too large to enable one to calculate accurately the wind variance from the height variance for synoptic purposes, but the relationships do allow climatological estimates (seasonal, etc.) to be made of the mean kinetic energy and of the size and amplitude of the dominant wave systems.

1. Introduction

KLEIN (1951) was one of the first to recognize that there must exist a high correlation between the variability of pressure and the variability of pressure gradient or wind. This correlation has recently been investigated, both theoretically and numerically, by JENKINSON (1956) and by BUELL (1954, 1957). Earlier studies have considered only standard deviations at a point; here we shall treat the problem of variability in space and time, separately and simultaneously, although the tests on actual data to be reported refer specifically to space variability.

2. Relations between space and time variances

The statistical measure of variation most commonly employed is the variance, or its square root, the standard deviation. We may discuss variation in time, at a point, variation in space, at a given time, variation in space for time-meaned quantities and variation in time for space-meaned quantities. There are thus four different standard deviations which may usefully be investigated, and we shall here carry out such an analysis, both for the vector standard deviation of geostrophic wind velocity (represented by σ with a defining subscript) and for the standard deviation of contour height (represented by S with a similar set of subscripts). It will be convenient to use a horizontal bar to denote a time average and square brackets to denote a space average; u and v will denote the x- and y-components of wind velocity (more strictly, of geostrophic velocity).

Consider first the vector standard deviation of wind velocity for variation in time (at a point), σ_4.

$$\sigma_4^2 = \sigma_{u_4}^2 + \sigma_{v_4}^2 = \overline{V^2} - V_r^2, \tag{1}$$

where V_r denotes the resultant velocity, the absolute value of the time-mean vector velocity. EADY (1951) pointed out that σ_4^2 was a quantity of fundamental importance in dynamic climatology, being a measure of the turbulent (large-scale, chiefly) kinetic energy. We shall return to this concept a little later.

Consider now the vector standard deviation of wind velocity for variation in space (at a single time), σ_1.

$$\sigma_1^2 = \sigma_{u_1}^2 + \sigma_{v_1}^2 = [V^2] - V_s^2, \tag{2}$$

where V_s denotes the space-mean vector wind velocity.

Consider now the vector standard deviation of space-mean wind velocities (in time), σ_2. For a large area, this will be a relatively small quantity.

$$\sigma_2^2 = \overline{V_s^2} - V_{rs}^2, \tag{3}$$

where V_{rs} denotes a space and time vector-resultant mean wind velocity.

Consider finally the vector standard deviation of time-mean wind velocities (over an area), σ_3.

$$\sigma_3^2 = [V_r^2] - V_{rs}^2. \tag{4}$$

Taking the space average of (1), and the time average of (2), and introducing (3) and (4), one may express the total variance of vector velocity in the alternate forms

$$\overline{[V^2]} - V_{rs}^2 = \overline{\sigma_1^2} + \sigma_2^2 = \sigma_3^2 + [\sigma_4^2]. \tag{5}$$

$\overline{[V^2]}$ is simply equal to twice the space and time average of the kinetic energy per unit mass. For a temperate-latitude area, the term V_{rs}^2 in (5) will make an appreciable contribution, but this will not be so for an area centred at the pole. On individual occasions for such a polar area, V_s would be small for quasi-zonal flow or for a quasi-symmetric cellular pattern; it would have an appreciable value, in fact, only with a pronounced trans-polar circulation.

Exactly analogous relations can be written down for the standard deviations of contour height, here denoted by S_1, S_2, S_3 and S_4.

$$\begin{aligned} S_1^2 &= [Z^2] - [Z]^2, \\ S_2^2 &= \overline{[Z]^2} - \overline{[Z]}^2, \\ S_3^2 &= [\overline{Z}^2] - [\overline{Z}]^2, \\ S_4^2 &= \overline{Z^2} - \overline{Z}^2. \end{aligned} \tag{6}$$

Once again the total variance, in both space and time, of contour height is

$$\overline{[Z^2]} - \overline{[Z]}^2 = \overline{S_1^2} + S_2^2 = S_3^2 + [S_4^2]. \tag{7}$$

3. Relations between velocity and height variances

Let us consider a cartesian grid, with grid spacings of δx and δy. For the time being, at least, we shall neglect variations over this grid of map magnification and of Coriolis parameter ($f = 2\Omega \sin \Phi$, where Ω = earth's angular velocity and Φ = latitude). Let us denote two immediately adjacent grid points by the subscripts p and q (separated by either δx or δy). Let us first analyze the problem treated by JENKINSON (1956) and BUELL (1954, 1957)—that of finding σ_4^2 in terms of S_4^2. Making the geostrophic assumption (denoting the ratio f/g by λ), we obtain from (1) and (6) a relation derived by JENKINSON (1956),

$$\lambda^2 \delta y^2 \sigma_{u_4}^2 = (S_{p_4} S_{q_4})^2 + 2 S_{p_4} S_{q_4} \{1 - r_4(\delta y)\}, \tag{8}$$

where $r_4(\delta y)$ denotes the correlation coefficient between Z_p and Z_q (a time correlation).

In terms of derivatives, the approach adopted by Buell (loc. cit.), (8) is equivalent to

$$\lambda^2 \sigma_{u_4}^2 = \left(\frac{\partial S_4}{\partial y}\right)^2 - S_4^2 \left[\frac{\partial^2}{\partial y^2} r_4(y)\right]_{y=0}, \tag{9}$$

where we have invoked l'Hospital's Rule for the final term.

Differentiation of the final set of (6) with respect to y and substitution into (9) give the relation

$$\lambda^2 \sigma_{u_4}^2 = -S_4^2 \frac{\left[\frac{\partial^2}{\partial y^2} r_4(y)\right]_{y=0}}{\{1 - R_4^2(Z, u)\}}, \qquad (10)$$

where R_4 (Z, u) indicates the correlation coefficient (in time, at a point) between contour height and the x-component of the velocity.

An analogous relation can be derived for $\sigma_{v_4}^2$, which, in combination with (10), gives

$$\lambda^2 \sigma_4^2 = -S_4^2 \left(\frac{\left[\frac{\partial^2}{\partial x^2} r_4(x)\right]_{x=0}}{\{1 - R_4^2(Z, v)\}} + \frac{\left[\frac{\partial^2}{\partial y^2} r_4(y)\right]_{y=0}}{\{1 - R_4^2(Z, u)\}} \right). \qquad (11)$$

According to JENKINSON (1956), the coefficient of S_4^2 in (11) is approximately a universal constant in space and time, throughout the troposphere. Without necessarily accepting this view (which we shall later show to be of dubious validity), we may write (11) in the form

$$\sigma_4^2 = \frac{k_4^2 S_4^2}{\sin^2 \phi}. \qquad (12)$$

Computations by BUELL (1954) of $k_4{}^2$ (see his figures 7, 8 and 9) indicate systematic geographical variations of this quantity, quite marked in polar areas, but rather less variation between troposphere and stratosphere than claimed by JENKINSON (1956).

The relation between σ_1^2 and S_1^2 (areal variances) can be found by carrying out the above analysis with space means rather than time means. Assuming that $\sin \phi$ has a small range or is only weakly correlated with S_1 or σ_1, one obtains

$$[\lambda^2] \sigma_1^2 = -S_1^2 \left(\frac{\left[\frac{\partial^2}{\partial x^2} r_1(x)\right]_{x=0}}{\{1 - R_1^2(Z, v)\}} + \frac{\left[\frac{\partial^2}{\partial y^2} r_1(y)\right]_{y=0}}{\{1 - R_1^2(Z, u)\}} \right), \qquad (13)$$

where the subscript 1 indicates a correlation over space, rather than in time. Without implying a constancy for the ratio of variances, let us set

$$\sigma_1^2 = \frac{k_1^2 S_1^2}{[\sin^2 \phi]}. \qquad (14)$$

If k_4 were invariant in space and time, then one would expect k_1 to have similar properties (and a similar value). The explicit formulations of k_4 and k_1 (see (11) and (13)) do not in themselves lend much support to such beliefs, nor do they indicate what would be the implications of such a quasi-constancy if real. Accordingly, it was decided to investigate a very simple contour model, to provide a useful basis for the discussion of various sets of experimental data.

4. Variance relations for a simple contour pattern

Let us now evaluate k_1^2 for a simple cellular type of contour pattern, defined by

$$Z = A \sin \mu x + B \sin \nu y + C, \qquad (15)$$

where $\mu = 2\pi/L_x$ and $\nu = 2\pi/L_y$. L_x and L_y are wavelengths, and A and B amplitudes, in the x- and y-directions, respectively. For simplicity we shall imagine that a given map contains an integral number of waves (or cellular systems) in both coordinate directions so that it is adequate to average any parameters required over one wavelength in each direction.

For this contour pattern,

$$S_1^2 = (A^2 + B^2)/2, \qquad (16)$$

$$[\sin^2 \phi] \sigma_1^2 = \frac{g^2 \pi^2}{2\Omega^2} \left(\frac{A^2}{L_x^2} + \frac{B^2}{L_y^2} \right). \qquad (17)$$

From (14), (16) and (17),

$$k_1^2 = \frac{g^2 \pi^2}{\Omega^2 (A^2 + B^2)} \left(\frac{A^2}{L_x^2} + \frac{B^2}{L_y^2} \right). \qquad (18)$$

In the case that $L_x = L_y = L$,

$$k_1^2 = \frac{g^2 \pi^2}{\Omega^2 L^2}. \qquad (19)$$

If $L_x \neq L_y$, we may define L as the effective or weighted wavelength by the above relation. This effective wavelength, L, is closely related to the "radius" of contour systems, R_h, introduced by BUELL (1954, 1957) through the extrapolation of $\partial^2 r(x)/\partial x^2$ to a zero correlation at the rate observed at $x = 0$. It may be shown, in fact, that

$$L = \pi R_h = 2\pi/m, \tag{20}$$

where m is the characteristic "wave number" utilized by THOMPSON (1957) in his interpretation of the meaning of the quasi-constant k_4 deduced by JENKINSON (1956). Incidently, with the particular contour system envisaged here, the correlation would actually drop to zero at a distance $L/4$. Thus, from (19) and (20),

$$k_1 = \frac{g\pi}{\Omega L} = \frac{g}{\Omega R_h} = \frac{gm}{2\Omega}. \tag{21}$$

In order to compute variances applicable to a point (time variances), one may assume a fixed speed and direction for contour system motion. It then follows directly that (16) applies equally well to $S_4{}^2$, (17) to $\sigma_4{}^2$ and (18) and (19) to $k_4{}^2$. Thus, for a reasonably homogeneous area for which space and time sampling are essentially equivalent, the two sets of statistical variance parameters should be numerically very similar. Contour-height variance, from (16), will be a direct measure of the root-mean-square "system amplitude", and the variance ratio, k, from (19), will be a direct measure of the effective "system wavelength".

It can be seen from (19) that k would be approximately constant from one map to another (k_1) or from one point to another (k_4) only if the effective wavelength remained the same, regardless of amplitude (i. e., intensity) changes. Thus, one might expect rather similar ranges of k from one season to the next in regions where the "size" of contour systems does not vary greatly from one season to another. Within any given season, it would be unreasonable to expect the effective wavelength to remain constant. In fact, one should expect a tendency for amplitude to increase as wavelength increases, leading to a more rapid increase of the intensity (and hence variability) of the contour-system than of the wind regime. In such a case, k would decrease as S increases.

One may adduce evidence for this latter belief from the quasiisotropic nature of wind-component variance, first demonstrated by Brooks et al. (1950) for variances in time, at a point. For the schematic contour model represented by (15), $\sigma_{u_1} = \sigma_{v_1}$ implies that $B\nu = A\mu$, or that $B/L_y = A/L_x$. The empirical evidence for isotropic wind behaviour implies that the ratio of amplitude to wavelength in the two component directions tends to remain constant and thus one would expect that readjustment of wavelength to amplitude changes (or vice versa) should be such that the ratio itself would not change greatly. Undoubtedly, the actual variation of σ_1 with S_1, from day to day (or of σ_4 with S_4 from point to point), is rather complex so that it would not be fruitful to examine more realistic contour models. Instead, one should carry out analyses on the actual atmosphere, and results of such analyses will be presented in the next two sections.

5. Experimental testing of variance relations

From the considerations of the previous section it appeared likely that the height variance (in space) of a pressure surface and the geostrophic wind variance would be significantly correlated but that the regression coefficient for a simple linear regression would not prove to be a universal constant. Although the concept is quite general (i.e., holds wherever the geostrophic assumption is valid), it was tested in the Arctic at 500 mb because the basic data were already available from a previous study.

The experiment consisted simply of computing the height and geostrophic wind variances for a number of sample days, and of determining the nature of their correspondence and the accuracy with which the wind variance could be estimated from the variance of height. A detailed account of the testing procedures, with data tabulations, has appeared elsewhere (GODSON and MACFARLANE, 1958). It should be noted that a close relationship would imply that the space-mean kinetic energy could also be estimated from contour-height variance (which latter parameter could then function as a highly efficient and readily calculable circulation index), since, from (2) and (14),

$$[V^2] = V_s^2 + \frac{k_1^2 S_1^2}{[\sin^2 \phi]}. \tag{22}$$

The source of data for this experiment was the 500-mb (0300 GMT) synoptic charts for

1955 drawn by the Department of Transport Arctic Forecast Team at Edmonton. In connection with another project (HARE, GODSON et al. 1957), arrays of contour height had been abstracted from these maps on a 13×13 point grid extending from the pole to 60° N (on the average). The values of S_1 for these arrays had also been calculated and only σ_1 had to be computed. To satisfy the differential relations of (13), $[\sin \phi] \sigma_1$ was formed for grid sizes of δ, $\sqrt{2}\delta$ and $2\delta (\delta x = \delta y = \delta)$, and then extrapolated analytically to a zero grid size.

The sample used in this study consisted of sixty days chosen to include all the seasons and the complete range of S_1. Because of the deliberate inclusion in the sample of the extreme values of S_1, a weighting factor had to be introduced so that the frequency distribution simulated the actual distribution of S_1 for 1955.

Scatter diagrams were made showing the relation between $[\sin \phi] \sigma_1$ and S_1, and between k_1 and S_1. Neither plot appeared to represent a linear correlation, but before attempting to fit a non-linear regression equation the data were converted to natural logarithms as it was thought most probable that the errors in computing S_1 and $[\sin \phi]\, \sigma_1$ were proportional to their magnitudes.

The new scatter diagram of the converted standard deviations (fig. 1) seemed to indicate a linear relationship between the two quantities and a straight line was fitted to the data, of the form

$$\ln [\sin \phi] \sigma_1 = a + b \ln S_1. \qquad (23)$$

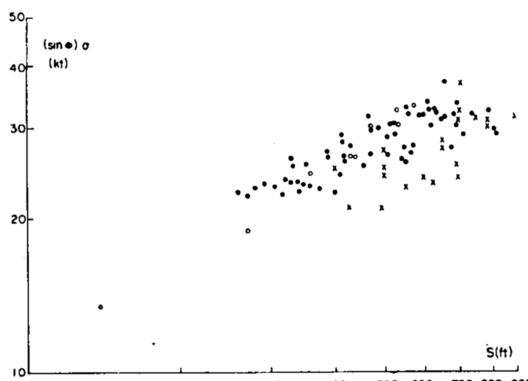

Fig. 1. Relation between contour-height standard deviation (S) and vector wind standard deviation (σ) at 500 mb at high latitudes (on logarithmic scales): closed circles—this study, open circles—JENKINSON (1956) and crosses—BUELL (1957).

Then, from (14),

$$k_1 = e^a S_1^{b-1}. \qquad (24)$$

The statistical parameters pertaining to the regression equation (23) are shown in Table 1.

Table 1. The linear regression of ln $[\sin \phi]\, \sigma_1$ on ln S_1

Statistic		Sample Value	5 % Confidence Limits	
			Lower	Upper
Regression Coefficient	b	0.403	0.306	0.500
Regression Intercept	a	1.768	1.394	2.141
Correlation Coefficient	r	0.736	0.590	0.836

Thus, from (24)

$$k_1 = 5.860\ S_1^{-0.597}. \qquad (25)$$

These results differ markedly from those of the previous studies (Buell and Jenkinson, loc. cit.) in which b was assumed equal to unity (so that k_4 became a constant). Jenkinson computed k_4 directly from eq. (12), whereas Buell computed both his regression coefficient and intercept but found that the latter was approximately zero. Because of the apparent discrepancies between the results, it seemed desirable to combine and reanalyse the data that Buell and Jenkinson had presented in their papers.

6. Comparison of space and time variances

As it was felt that any valid comparison of these space and time statistics should be made with overlapping data, only the values of σ_4 and S_4 at 500 mb and from the northern stations ($\phi \geq 45°$) were used. The analysis described in section 5 was repeated and the results are given in Table 2 (see figure 1 for data points).

Table 2. The linear regression of ln $(\sin \phi)\sigma_4$ on ln S_4

Statistic		Sample Value	5 % Confidence Limits	
			Lower	Upper
Regression Coefficient	b	0.426	0.300	0.533
Regression Intercept	a	1.580	1.078	2.081
Correlation Coefficient	r	0.788	0.578	0.895

From (24)

$$k_4 = 4.853 \ S_4^{-0.574}. \tag{26}$$

The confidence limits are somewhat wider than before (smaller sample) but otherwise these values are similar to those obtained from the space variances (see Table 1).

Although the high correlation between the space variance of contour height and the space variance of the geostrophic wind is the first conclusion of this study, it is not an unexpected one in the light of the underlying theory and the results obtained by Buell and Jenkinson; on the other hand, the most interesting result is the nature of the relationship between these two quantities.

The general form of the dependence of wind variance on contour height variance can be expressed as

$$\sigma \alpha S^b, \ 0.3 \leq b \leq 0.5, \tag{27}$$

which is applicable to either the space or time values. At least a superficial explanation of this proportionality can be made on the basis of the contour model discussed in section 4. That is, those pressure patterns in which there is a pronounced coupling of the wave amplitude with the wavelength dominate over the patterns of variable amplitude but more or less constant wavelength. This coupling would most probably occur in long-wave patterns, such as those of the circumpolar westerlies, rather than in short-wave disturbances which are more likely to be of roughly uniform size though of widely varying intensity.

Moreover, it appears from the similarity of (25) and (26) that the space and time statistics are both sampling the same information, and the form of (27) suggests that this information is concerned with the largest features of the pressure patterns. Although this conclusion ($k_1 = k_4$) had appeared plausible, the empirical evidence seemed to be against it until the time statistics were reanalysed.

Naturally, even in a region of long-wave disturbances, the ratio of amplitude to wavelength would not be constant from day to day or from place to place; hence the considerable scatter of points in figure 1 is only to be expected. That the standard error of estimate of σ_4 is noticeably larger than that of σ_1 is probably due, in part at least, to the location of the samples. The space grid is embedded at the centre of the circumpolar vortex, which is made up of a core of small cellular disturbances surrounded by long-wave patterns; hence the values of S_1, σ_1 and k_1 probably represent some sort of weighted average of the values that would have been obtained by sampling the two regimes separately. This might tend to damp down the range of residuals about the regression line. As opposed to this, regional differences would be emphasized by point sampling in time. BUELL (1954) has shown that in fact there are considerable variations in the size of pressure systems from one area to another. Thus the time sample used here is somewhat unsatisfactory because the stations not only tend to be near centres of action but they do not sample more than the fringe of the space grid (GODSON and MACFARLANE, 1958, fig. 10).

In the face of these possible sources of difference it is encouraging to find that there is no significant difference between the relationships in space and time of the standard deviations of height with the standard deviations of wind. This has a bearing on the remark by BUELL (1957) that the statistical theory of turbulence (BATCHELOR, 1953) leads to the relation: $\sigma^2 \alpha S$. Since homogeneous turbulence, by definition, is such that space and time sampling give identical results, one might think it possible to consider the large scale features of the 500-mb circulation as being two-dimensionally isotropic homogeneous turbulence. However, it is pertinent to note that the isotropic turbulence prediction ($\sigma^2 \alpha S$) is formulated for a system in which Coriolis forces can be ignored. This is clearly not true in the present case, so that the apparent agreement between the σ—S relations in the two cases should be regarded as largely accidental.

Equations (25) and (26) are of considerable climatological value. From these relations it is possible to obtain estimates, albeit crude, of the standard vector deviation of the wind, the mean kinetic energy and the r.m.s. amplitude and effective wavelength of dominant pressure systems, all from an easily obtained parameter—the variance of contour height. For not only can S_1 be readily computed from constant pressure charts, but it is a standard parameter developed in all types of numerical specification techniques.

In the year 1955 (using the Wadsworth specification data described in section 5) the annual mean value of S_1 (i.e., the r.m.s. amplitude from (16)) was almost 500 ft., representing a mean radius of some 750 n. miles (12½° of latitude). This corresponds well with the mean size of Arctic pressure systems obtained by direct measurement from 500-mb charts.

Acknowledgement

The research reported in this document has been sponsored in part by the Geophysics Research Directorate of the Air Force Cambridge Research Center, Air Research and Development Command, under Contract No. AF 19 (604)-1141. Participation by Dr. W. L. Godson in this research is with the permission of the Director, Meteorological Branch, Department of Transport, Canada.

The authors wish to acknowledge the interest and encouragement provided by Prof. F. Kenneth Hare, Project Director, Arctic Meteorology Research Group, McGill University.

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