THE OCEANIC THERMOCLINE

Authors' Foreword to the Two Following Papers on the Oceanic Thermocline

Each of the two following papers attempts to provide a theoretical framework for explaining the oceanic thermocline, and the associated thermohaline circulation of the ocean. They were developed independently, and then the authors exchanged copies of their papers. As they stand, the two theories are not compatible; but the results of each are similar in certain respects, and apparently resemble the actual ocean insofar as we actually know it. Because of the basic differences in the two theories we have decided not to try to combine them into a single paper, but to publish them together in the same number of this journal, so that they should both appear in the literature simultaneously, and so that the marked contrasts in the basic formulation should be immediately evident to the reader.

Allan Robinson, Henry Stommel, Pierre Welander

The Oceanic Thermocline and the Associated Thermohaline Circulation[1]

By ALLAN ROBINSON[2] and HENRY STOMMEL

Woods Hole Oceanographic Institution, Woods Hole, Massachusetts

(Manuscript received November 15, 1958)

Abstract

A study is made of the thermal structure of an ocean which is dynamically geostrophic, bounded by an eastern coast, and driven by an imposed surface temperature distribution and wind-stress. The heat equation contains vertical diffusion in the form of a virtual eddy mixing parameter and non-linear terms of vertical and north-south advection. Comparison of the results with the North Atlantic shows an agreement with the observed temperature distribution in the thermocline region, including the deepening of isotherms in mid-latitudes. The value of the vertical component of velocity at the bottom of the thermocline is predicted, and a numerical value for the eddy-conductivity obtained.

1. Introduction

Ninety years ago the question of whether the wind-stresses or thermally produced differences of density are the predominant cause of oceanic circulation was a subject of public debate.[3] Since that time, this central issue of physical oceanography has been treated with more caution, and is only indirectly alluded to in present-day textbooks. That the

[1] Contribution No. 1003 from the Woods Hole Oceanographic Institution.

[2] Present address, Dept. of Physics, Harvard University. Part of the material presented here has been included in a doctoral thesis presented to the Dept. of Physics, Harvard University.

[3] The exchange of letters in NATURE during the 1870's between W. B. Carpenter and James Croll shows how futile simple verbal arguments can be in discussing such issues. The reader with a morbid interest in fallacious verbal theories may find it entertaining to look over the work of the English eccentric William Leighton Jordan, "The Ocean, its tides and currents and their causes", Longman's Green and Co., London, 1873.

vertical integral of mass transport of the ocean can probably be attributed to the wind-stress acting upon the surface was established by SVERDRUP in 1947. This important concept, and its further elaboration in the hands of MUNK (1950), HIDAKA (1950), and others, are described in a survey article by STOMMEL (1957). Computation of a vertical integral, however, gives no idea of the distribution of a property in the vertical, whereas the geostrophic calculation of currents from observed hydrographic station data gives only the vertical distribution of the velocity vector, but not its integral. Moreover, the vertical integration completely cancels the contribution of the thermohaline circulation. Consequently, there was a tendency, starting in 1950, to minimize the possible role of the thermohaline circulation.

The first attempts to formulate a model containing an active thermal-convective process were

(1) an important Russian paper by P. S. LINEYKIN (1955)
(2) an unpublished doctoral dissertation by FOFONOFF (1954)
(3) chapter XI in a book on the Gulf Stream, written in 1954—5 by STOMMEL (published in 1958).

Studies (1) and (2) had no meridional boundary —the key feature of the Sverdrup wind-driven model—and so were not directly applicable to actual oceanic basins. Study (3) treated only the horizontal patterns of flow that would ensue in a meridionally bounded basin on the assumption of a fixed vertical mass flux at mid-depth, and did not treat the important vertical heat flux portion of the problem.

Later, VERONIS and STOMMEL (1957) and LINEYKIN (1957) were able to modify study (1) so that the important influence of the variation of the Coriolis parameter with latitude could be incorporated; and obtained expressions for the depth of the thermocline which seemed more related to reality. On the basis of these models STOMMEL (1958) indicated how they might be applied to give a rather detailed picture of the patterns of flow in the thermohaline circulation of the world-ocean. Nevertheless, as explained in the survey article (STOMMEL, 1957) neither of these models imitates the actual oceanic density distribution very closely; hence it was desirable to construct a theoretical model of the oceanic thermohaline circulation with an eastern wall to inhibit zonal flow, driven by a meridional temperature gradient fixed at the surface, and without using the totally unrealistic large vertical temperature gradient imposed by both LINEYKIN and VERONIS and STOMMEL on the models mentioned above for the sake of linearization of the convection equation.

The present paper is an attempt to construct such a model, and we believe it is a substantial improvement upon previous studies. However, it contains—as they also do—a parametric treatment of the mixing processes embodied in an "eddy thermometric conductivity" parameter $\varkappa$. This parameter is assumed to be uniform and constant over the entire ocean basin —in contradistinction to other studies which attribute the oceanic thermocline essentially to variations in $\varkappa$ (for example, DEFANT, 1936; MUNK and ANDERSEN, 1948). Thus we envisage the ocean as being slowly and evenly stirred by some physical process which we cannot specify, and the necessity for so doing is a measure of how far short we are of actually being able at present of giving a complete physical account of the thermal convective circulation of the ocean.

2. Formulation

We consider an ocean thermally driven by an imposed north-south temperature distribution on the surface. The constraint of rotation is represented by a Coriolis parameter varying linearly with latitude. The ocean is taken to be infinitely deep and bounded only by an east coast. We inquire into the non-linearly coupled temperature and velocity fields in the steady state.

Let (x, y, z) represent the east-west, north-south, and vertical respectively, with (u, v, w) the corresponding velocity components. The origin is fixed in the east coast at mid-latitudes. The region of interest is $x < 0$, $y_1 < y < y_2$, $z < 0$. The fluid system to be considered is defined explicitly by the following assumptions.

The equation of state of the fluid varies linearly with temperature only. Furthermore, the coefficient of thermal expansion, α, is considered zero except when coupled with g, the acceleration of gravity. In the equations of motion we neglect all viscous and inertial terms, balancing by the pressure gradient the gravitational force in the vertical and the Coriolis accelerations in the horizontal. In summary we use purely geostrophic dynamics together with the Boussinesq approximation. The equations of motion and of mass continuity take the form

$$f\varrho_0 v - \frac{\partial p}{\partial x} = 0 \tag{1}$$

$$f\varrho_0 u + \frac{\partial p}{\partial y} = 0 \tag{2}$$

$$g(\varrho_0)(1 - \alpha T) + \frac{\partial p}{\partial z} = 0 \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

where $f = f_0 + \beta y$, the Coriolis parameter, ϱ_0 is the mean density, p the pressure, and T the temperature.

In the heat equation we assume a balance between diffusion in the vertical by a constant eddy-conductivity $\varkappa$, and vertical and north-south advections,

$$\varkappa \frac{\partial^2 T}{\partial z^2} - w \frac{\partial T}{\partial z} - v \frac{\partial T}{\partial y} = 0 \tag{5}$$

The choice of $\varkappa$ will be discussed later. The neglect of $u \dfrac{\partial T}{\partial x}$ can be justified only in terms of boundary conditions and anticipated results. The z boundary conditions on the temperature will be to specify T as a function of y only at $z = 0$, and to require that T approach a constant, zero, asymptotically as $z \to -\infty$. The horizontal velocities, both of the same order of magnitude, will be largest at the surface and go to zero asymptotically. Thus, near the surface, u, v, and $\dfrac{\partial T}{\partial y}$ have their largest values while $\dfrac{\partial T}{\partial x}$ is specified as zero. Clearly $v \dfrac{\partial T}{\partial y}$ will dominate over $u \dfrac{\partial T}{\partial x}$. The inclusion of a western boundary would, of course, change the above considerations locally.

The fields of primary interest are T and w; two equations in these alone can be obtained. By cross-differentiation of (1) and (2) and use of (4), we obtain

$$\beta v - f \frac{\partial w}{\partial z} = 0 \tag{6}$$

Cross-differentiation of (1) and (3) yields

$$f \frac{\partial v}{\partial z} - \alpha g \frac{\partial T}{\partial x} = 0 \tag{7}$$

Inserting v from (6) into (5) and (7) yields the desired equations,

$$\varkappa \frac{\partial^2 T}{\partial z^2} - w \frac{\partial T}{\partial z} - \frac{f}{\beta} \frac{\partial w}{\partial z} \frac{\partial T}{\partial y} = 0 \tag{8}$$

and

$$\frac{\partial^2 w}{\partial z^2} - \left(\frac{\alpha \beta g}{f^2}\right) \frac{\partial T}{\partial x} = 0 \tag{9}$$

The boundary conditions imposed at $z = 0$ are $T = T_0 + T_1 y$, and $w = 0$; as $z \to -\infty$, $T \to 0$, and w remains finite, i.e. $w \to w_\infty(x, y)$. The boundary condition at $x = 0$ is taken, for mathematical convenience, as $T = 0$. This assures no flow through the plane $x = 0$, but is otherwise physically unrealistic. However, in no real fluid or ocean would the above equations describe the region near such a boundary; therefore our detailed results very near the coast will not be valid. Anticipating carrying the y-dependence parametrically, we reserve discussion of this boundary condition.

3. The Similarity Transformation

T and w are specified at $z = 0$ and approach arbitrarily close to their asymptotic values along some curve $z = f(x)$. In related problems it is often found that at a given x_0 the approach of the fields to their asymptotic values depends only on the ratio of the actual depth to the value $f(x_0)$, of the curve at this point (GOLDSTEIN, 1938). The formal analogy to these boundary layer problems occurs because only vertical mixing is allowed, so that only higher order derivatives with respect to z are present. Formally, we seek separated solutions of the form $T = G(x)\vartheta(\xi)$, $w = H(x)\omega(\xi)$, where $\xi = zF(x)$, and all functions are tacitly functions of y. This transformation is also convenient for satisfying the z boundary conditions, e.g., at $z=0$, $\xi=0$ and T becomes a function of x. In the following a prime denotes differentiation with respect to ξ, and a subscript y, differentiation with respect to y.

Taking x and z derivatives in terms of x and ξ, and inserting these in (8) and (9), we obtain

$$\varkappa F\vartheta'' - H\left[\omega\vartheta' - \left(\frac{f}{\beta}\right)\omega'\vartheta_y\right] = 0 \quad (10)$$

$$HF^2\omega'' - \left(\frac{\alpha\beta g}{f^2}\right)\left[G'\vartheta + \frac{GF'}{F}\xi\vartheta'\right] = 0 \quad (11)$$

The condition for separability of (10) in x and ξ is merely that H be a constant times F; this constant can be absorbed into ω, so we require $F=H$. For equation (11) to separate all three terms must have the same x-dependence. The second and third terms require that G'/G be a constant times F'/F, or that G be an arbitrary power of F. The first and second terms then require that F be some power of $x+c$ where c is a constant.

In summary, under the general transformation

$$\xi = z(x+c)^k, \quad T = (x+c)^{3k+1}\vartheta(\xi),$$
$$w = (x+c)^k\omega(\xi)$$

(8) and (9) take the form

$$\varkappa\vartheta'' - \omega\vartheta' - \left(\frac{f}{p}\right)\omega'\vartheta_y = 0 \quad (12)$$

$$\omega'' - \left(\frac{\alpha\beta g}{f^2}\right)[(3k+1)\vartheta + k\xi\vartheta'] = 0 \quad (13)$$

The constants k and c are determined by the particular boundary conditions. To make the surface temperature independent of x, we choose $k = -\frac{1}{3}$. The choice of $c=0$ gives $\xi = zx^{-1/3}$; $\xi\to\infty$ for both $z\to-\infty$ and $x\to 0$, and the east coast temperature will automatically correspond to the bottom temperature.

4. Order of Magnitude Considerations

Equations (12) and (13) are still too complicated to be solved exactly. However, from them we can determine certain gross properties of the temperature and velocity fields: their general structure, the asymptotic value of w, and the scale of ξ within which all asymptotic values are essentially obtained. Moreover, it would not be much more meaningful to have the details of the exact solutions because the mathematical model which we are using is itself not detailed, and also because the results are to be compared with the averaged observations of the real ocean.

In seeking unknown orders of magnitude, it is convenient to introduce known magnitudes in terms of slightly different parameters. Let $\eta = 10^8 y$, $\gamma(y) = \frac{f(y)}{\beta}\cdot 10^{-8}$, and $\varepsilon(y) = \frac{\alpha\beta g}{3f^2(y)}$. η and γ are of order unity while ε is of order 10^{-6}. We expect $\varkappa$ also to be order unity. To determine how ε affects the amplitude of ω and scale of ξ we write $\zeta = \varepsilon^a\xi$ and $W = \varepsilon^{-b}\omega$. We assume that the scale of ζ and amplitude of W are of order unity, and further that ϑ and W are smooth functions of ζ, i.e. the functions and all their derivatives are also of order unity.

Under these assumptions, (12) and (13) become, with a prime now denoting differentiation with respect to ζ

$$\varepsilon^{2a}\varkappa\vartheta'' - \varepsilon^{a+b}W\vartheta' - \varepsilon^{a+b}\gamma W'\vartheta_\eta = 0 \quad (14)$$

$$\varepsilon^{2a+b}W'' + \varepsilon\zeta\vartheta' = 0 \quad (15)$$

Requiring that the dynamical equation (15) always retain both terms gives $2a+b=1$. In the heat equation (14), we notice that both advections are necessarily the same order of magnitude. Insisting that vertical eddy-diffusion be comparable to the advections yields $a=b=\frac{1}{3}$. The parameter ε now disappears from (14) and (15), which now contain only terms of roughly unit magnitude. From the similarity transformation we have obtained the dependence of the asymptotic value of w and the scale depth on x, from the above considerations the order of magnitude and major part of the y dependence of these quantities.

5. Parametric Linearization and Solution

The mathematical difficulties remaining are inherent in the two nonlinear terms of (14). To obtain approximate results, we shall replace these actual terms by linear terms, a technique which has found frequent use in convection problems (LEWIS and CARRIER, 1949). In the last term of (14) we allow for convection with an average temperature gradient instead of the actual ϑ_η. In the next to the last term we allow for convection by an average vertical velocity instead of the actual W. The averages are not to be taken over the full range of ζ, but only from the surface to a depth L at which asymptotic values are obtained arbitrarily closely. For convenience we define L explicitly as the depth at which the fields have approached to within e^{-2} of their asymptotic values.

At this point the difficulty arises that the average values of ϑ_η and W are unknown. However, we do know that ϑ_η ranges from T_1 at the surface to essentially zero at the depth L. We therefore crudely take $T_1/2$ as the average ϑ_η, although the true average will vary in y with the shape of ϑ. However, we have absolutely no information about W. Therefore, we shall allow the unknown average value of W to enter the equations and parameterize the solutions. Then, consistently taking the average of the parameterized solution actually obtained will yield an algebraic equation in the average W. Finally, the solution of the algebraic equations, in terms of the external parameters, is inserted back into the solutions for ϑ and W.

Replacing (14) and rewriting (15), the final linear set of equations is

$$\varkappa\vartheta'' - \varphi\vartheta' - \gamma\frac{T_1}{2}W' = 0 \tag{16}$$

$$W'' + \zeta\vartheta' = 0 \tag{17}$$

where

$$\varphi = \frac{1}{L}\int_0^L W(\zeta)\,d\zeta \tag{18}$$

A defining equation for L in terms of φ and the external parameters will be immediately available from the solutions of (16) and (17).

Exact solutions of (16) and (17) for ϑ and W have been found. These solutions consist of exponentials times Airy functions of purely complex argument, containing φ in an involved manner. The solution ϑ and W have the desired properties, i.e. they can satisfy all the boundary conditions. It would be possible but tedious to calculate φ from the exact solutions. At this point, with the knowledge of the exact solutions, and with the lack of real meaning of the detailed form of the solutions W and ϑ, we again proceed approximately.

The quantities of primary interest are the depth L and the value of W_∞. To find these properties of the solutions, we shall use forms for W and ϑ which exactly satisfy the boundary conditions and exactly satisfy the first integrals of the differential equations, i.e. have the exact average properties. Most simply, we take

$$\vartheta = \vartheta_0 e^{-\frac{2}{L}\zeta} \tag{19}$$

$$W = W_\infty\left(1 - e^{-\frac{2}{L}\zeta}\right) \tag{20}$$

Multiplying (16) and (17) by $d\zeta$, integrating from zero to infinity,

$$-\varkappa\vartheta'(0) + \varphi\vartheta_0 - \frac{\gamma T_1}{2}W_\infty = 0 \tag{21}$$

$$W'(0) + \int_0^\infty \vartheta\,d\zeta = 0 \tag{22}$$

In (21) and (22) the derivatives and integral can be substituted from the forms (19) and (20). From (20) we also have simply $\varphi = \frac{W_\infty}{2}$.

Solving the algebraic equations, we obtain

$$\left.\begin{aligned} L &= 2\{(2\varkappa)[\vartheta_0 - T_1\gamma]^{-1}\}^{\frac{1}{2}} \\ &= 2\left\{(2\varkappa)\left[T_0 + \frac{T_1\times 10^{-8}f_0}{\beta}\right]^{-1}\right\}^{\frac{1}{2}} \end{aligned}\right\} \tag{23}$$

which is independent of y, and

$$W_\infty = -\vartheta_0 L^2/4 \tag{24}$$

The above results could of course be improved by taking higher moments of eq. (16) and (17) and introducing additional constants into the forms (19) and (20).

The approximate forms for the temperature and vertical velocity are

$$T = \vartheta_0(y)\exp\left\{-\left[\frac{1}{2\varkappa}\frac{\alpha\beta g}{3f^2(y)}\left(T_0 + \frac{T_1 f_0}{\beta}\times 10^{-8}\right)\frac{1}{x}\right]^{\frac{1}{3}}z\right\} \tag{25}$$

$$w = \vartheta_0(y)\left[\frac{\alpha\beta g}{3f^2(y)}\left(\frac{2x}{T_0 + \frac{T_1 f_0}{\beta}\times 10^{-8}}\right)^2 \frac{1}{x}\right]^{\frac{1}{3}} \cdot (1-\exp\{\ \}) \tag{26}$$

6. Finite depth

It is apparent that in mid-latitudes the thermocline is so shallow that it occupies only a small fraction of the actual ocean depth, and that the assumption of infinite depth is justified in making the calculations of ϑ and W in the thermocline region itself. James Crease showed us in private correspondence concerning an earlier form of the thermocline model that this was likely to be the case. But the real ocean in fact has a bottom, and if this bottom were flat the vertical component of velocity W must vanish there. Thus we regard the quantity W_∞ as being a measure of the vertical velocity just beneath the thermocline (that is, at infinity so far as the thermal boundary layer is concerned). The form of the similarity transformation does not permit us to fix the bottom at z = constant; instead, it is necessary to put the bottom at $\zeta = \zeta_b$ where $\zeta_b \gg L$, a very mild compromise. Evidently we can still use the form of solution for temperature given by equation (19) and need modify equation (20) only as follows

$$W = W_\infty\left(1 - \zeta/\zeta_b - e^{-\frac{2}{L}\zeta}\right) \tag{20'}$$

where the quantities $e^{-\frac{2}{L}\zeta_b}$ are so small at $\zeta = \zeta_b$ that they are truly negligible ($\sim e^{-10}$) and we can assert that these solutions very closely satisfy the boundary conditions $W = 0$, $\vartheta = 0$ at the bottom $\zeta = \zeta_b$. In the region of the thermocline the quantity $1 - \zeta/\zeta_b$ is so nearly unity that the formal results given in equations (21) and (22) and (23) are not changed except that in (21) the term $\frac{-\gamma T_1}{2} W_\infty$ will vanish altogether unless we restrict this approximate form of ϑ_η, $T_1/2$ to depths shallower than $\zeta = L$, and set it equal to zero for $\zeta > L$. This is quite reasonable to do under the circumstances. Thus the computation of L and φ is not affected by the presence of a bottom at a depth of several times the depth of the thermocline and the temperature field is essentially unaffected throughout the ocean. The only modification of the vertical distribution of the vertical component of velocity W is that instead of approaching a constant asymptotic value W_∞ at great depths it approaches a maximum value just beneath the thermocline, which we find convenient to call W_∞ nevertheless because this is still infinite depth so far as the thermocline depth is concerned, and then varies linearly from this maximum just under the thermocline to zero at the bottom. However, so far as horizontal components of velocity below the thermocline are concerned, the presence of a finite depth has a major effect: it gives them a finite amplitude (in the case of infinite depth they vanish beneath the thermocline). They are calculated from equation (6) the terms of which no longer tend to zero under the thermocline.

The depth of no meridional motion is at the depth at which $v = 0$ or at $W' = 0$, hence at $\zeta = \frac{L}{2} \ln \frac{2\zeta_b}{L}$. Since throughout most of the oceans in mid-latitudes $2\zeta_b/L \geq 5$, the depth of no meridional motion (or reference level for "dynamic calculations") is $\geq L$, or quite distinctly at the bottom of the thermocline. In the absence of wind-stress at the surface, W must vanish at both top and bottom of the ocean, so that the vertically integrated value of v, by equation (6) must vanish. This is consistent with Sverdrup's earlier wind-driven model. The thermohaline circulation is entirely an internal mode and does not contribute to vertical integrals of horizontal velocity components—but it is important in trying to understand the oceanic circulation just the same.

7. Inclusion of wind-stress

The dynamical equations (1) and (2) are non-viscous. For the wind stress to work upon the ocean it is necessary to allow viscosity to play a role somewhere, and this has generally been done by a boundary-layer analysis of the form of Ekman. Experience suggests that the Ekman viscous boundary layer is confined to an upper one hundred meters in the ocean. At any point in the ocean the total vertically integrated mass transport in the Ekman layer is to the right (in Northern hemisphere) of the direction of the applied wind-stress, τ, and of magnitude τ/f. Since both τ and f vary as functions of position over the ocean, the mass transport of the Ekman layer also varies from place to place. If we form the horizontal divergence of the Ekman layer, we find that this implies by equation (4) a vertical component of velocity at the bottom of the Ekman layer of an amount

$$w_e = -\frac{\partial}{\partial y}\left(\frac{\tau_x}{f}\right) + \frac{\partial}{\partial x}\left(\frac{\tau_y}{f}\right) \tag{27}$$

In regions of divergence of the Ekman layer, this vertical velocity is directed upward, and we need merely adopt this value instead of the value $W=0$ in forming the function given in equation (20). In regions where the vertical component of velocity beneath the Ekman layer is downward—as in the centers of the mid-latitude oceanic current gyres—the vertical velocity actually must pass through zero as a function of depth before it reaches the upward directed value W_∞ just beneath the thermocline. Since, in forming the parameter φ we take a vertical average of W it would be inappropriate to try to treat this case by proceeding as before.

The two cases are treated separately:
Case 1: $w_e < 0$, $W_e > 0$ (the Ekman layer is divergent; z is positive upward, ζ is positive downward). The forms of the solution (19) and (20) are rewritten as follows

$$\vartheta = \vartheta_0 e^{-\frac{2}{L}\zeta} \tag{19''}$$

$$W = W_\infty\left(1 - a e^{-\frac{2}{L}\zeta}\right) \tag{20''}$$

$$\text{At } \zeta = 0, \quad W = W_\infty(1-a) = W_e \tag{28}$$

hence $a < 1$. The definition of φ is now

$$\varphi = \frac{1}{L}\int_0^L W(\zeta)\,d\zeta = W_\infty\left(1 - \frac{a}{2}\right) \tag{18''}$$

The first integrals are now of the form

$$-\varkappa\vartheta'(0) + \varphi\vartheta_0 + \frac{\gamma T_1}{2}(W_e - W_\infty) = 0 \tag{21''}$$

$$W'(0) + \int_0^\infty \vartheta\, d\zeta = 0 \tag{22''}$$

and with the indicated substitutions the following relations obtain

$$\frac{2\varkappa}{L}\vartheta_0 + W_\infty\left(1 - \frac{a}{2}\right)\vartheta_0 + \frac{\gamma T_1}{2}(W_e - W_\infty) = 0$$

$$\frac{2aW_\infty}{L} + \frac{L}{2}\vartheta_0 = 0$$

The definition of a is given by (28).

In these equations $\varkappa$, r, T_1, W_e, ϑ_0 are regarded as known, the quantities a, L, W_∞ are to be found. In this case the value of L is given by the cubic

$$\frac{L^3}{8}(\vartheta_0 - \gamma T_1) - LW_e = 2\varkappa \tag{23''}$$

instead of the simple cube root (23). Once having obtained this we find

$$W_\infty = W_e - \frac{L^2\vartheta_0}{4} \tag{24''}$$

Case 2: $w_e<0$, $W_e>0$ (the Ekman layer is convergent). Here there is a reversal of the vertical component of velocity, W, with depth, and therefore we must divide the ocean into two layers: one from $\zeta=0$ where $W=W_e$ to $\zeta=h$ where $W=0$; and the other from $\zeta=h$ to $\zeta=\infty$ where $W=W_\infty$ which we treat exactly as before. For our purposes, if we define the unknown temperature at h as ϑ_1 we can write the temperature and vertical velocity in the lower layer as:

$$\vartheta = \vartheta_1 e^{-\frac{2}{L}(\zeta-h)} \tag{19'''}$$

$$W = W_\infty\left(1 - e^{-\frac{2}{L}(\zeta-h)}\right) \tag{20''}$$

The quantity h is also unknown, of course, however, we can write the quantities L and W_∞ in terms of ϑ_1 by equations (23) and (24). The only change here is that we can probably drop the advective term γT_1 in the expression for L, because we will incorporate the meridional advection mostly in the upper layer. As we have already seen, this term does not play a major quantitative role in the results. If the upper layer is not very thick a very rough approximation of the vertical velocity is given by a linear relation (see numerical computations, STOMMEL (1956)).

$$W \cong W_e(1 - \zeta/h) \qquad (29)$$

The average of W from $\zeta = 0$ to $\zeta = h$ is obviously $W_e/2$ and the vertical derivative W_e/h; moreover the approximate form of $\partial T/\partial y$ in the upper layer is T_1, so that we now can insert these approximations in the non-linear temperature equation and obtain (we can obtain a more accurate form for W later by solving the dynamical equation (22))

$$\varkappa\vartheta'' - \frac{W_e}{2}\vartheta' = -\gamma\frac{W_e}{h}T_1 \qquad (30)$$

The solution of this equation is of the form

$$\vartheta = C_1 e^{\frac{W_e\zeta}{2\varkappa}} + C_0 + 2\gamma T_1\left(\frac{\zeta}{h}\right) \qquad (31)$$

where C_1 and C_0 are constants of integration. The temperature must be equal to ϑ_0 at $\zeta = 0$ and to ϑ_1 at $\zeta = h$; therefore

$$C_1 + C_0 = \vartheta_0 \qquad (32)$$

$$C_1 e^{\frac{hW_e}{2\varkappa}} + C_0 = \vartheta_1 - 2\gamma T_1 \qquad (33)$$

Also, the temperature gradients at $\zeta = h$ must equal each other in both layers

$$C_1 e^{\frac{hW_e}{2\varkappa}} + \frac{2\gamma T_1}{h} = -\vartheta_1\frac{2}{L} \qquad (34)$$

and so must the vertical derivative of the vertical velocity component:

$$-\frac{W_e}{h} = \frac{2W_\infty}{L} \qquad (35)$$

Eliminating the constants of integration, C_0 and C_1 and also W_∞, we obtain three equations in which there are three unknowns h, L, and ϑ_1:

$$\left(1 - e^{\frac{-hW}{2\varkappa}}\right)\frac{2\varkappa}{W_e}\left(-\frac{2\gamma T_1}{h} - \frac{2\vartheta_1}{L}\right) + $$
$$+ \vartheta_0 + 2\gamma T_1 - \vartheta_1 = 0 \qquad (36)$$

$$\frac{W_e}{h} = \frac{L\vartheta_1}{2} \qquad (37)$$

$$L = 2(2\varkappa/\vartheta_1)^{\frac{1}{2}} \qquad (38)$$

One of these is, unfortunately, transcendental so that a general solution cannot be obtained, but different methods will be appropriate for different values of the parameters, involved. Rather than delving into a lot of algebra, we would like to point out that when dealing with the application of these ideas to the ocean, we do not really know $\varkappa$ to begin with. Instead we form an estimate of h, L, and ϑ_1, from the shape of the vertical temperature sounding. We obtain a rough estimate of W_e from charts of wind-transport of the surface layers, and then we do not use equations (36), (37), (38) in the same way. Equation (38) is used to obtain an estimate of $\varkappa$; and equations (36) and (37) can be used as a consistency check on the applicability of the model to the ocean. We know all the quantities that go into them, and they ought to be satisfied if the theory is applicable. We can then use equation (35) to calculate the vertical velocity just beneath the thermocline, W_∞. A further point: because of the nature of the similarity transformation, it is not easy to introduce vertical velocity at the surface, w_e, with an arbitrary x dependence: in fact we are restricted to w_e distributions of the form $x^{-1/3}$.

8. Application to a real ocean basin

The fact that the field of density calculated from equation (25) superficially resembles that naturally occurring in the ocean is demonstrated in figures 1 and 2. Figure 1 is a dimensionless perspective diagram of the solution (25) with an imposed surface temperature distribution of the form

$$\vartheta_0(y) = 20°\mathrm{C} - 25°\mathrm{C} \times y' \quad L_y = 10^8\ \mathrm{cm}$$

where we have introduced the non-dimensional lengths x', y' by the relations $x = L_x x'$, $y = L_y y'$, $z = Dz'$. L_x and L_y are dimensional constants defining the scales of the phenomenon, and chosen so as to make the numbers x', y' be in a range near unity. Thus the actual formula used in constructing the diagram was

$$T = (20 - 2.5\eta)\exp\left\{-H\frac{z'}{x'^{\frac{1}{3}}}\right\}$$

$$L_x = 5 \times 10^8\ \mathrm{cm}$$
$$D = 10^5\ \mathrm{cm}$$

where

$$H = \left[\frac{\alpha\beta g}{6\varkappa f^2}(20 + 12.5)\frac{D^3}{L_x}\right]^{\frac{1}{3}}$$

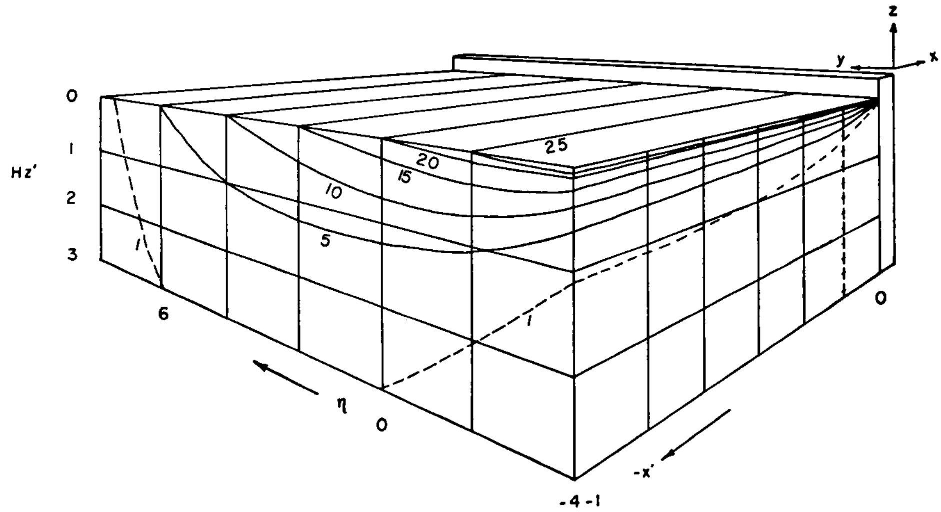

Fig. 1. Theoretical temperature field (see text).

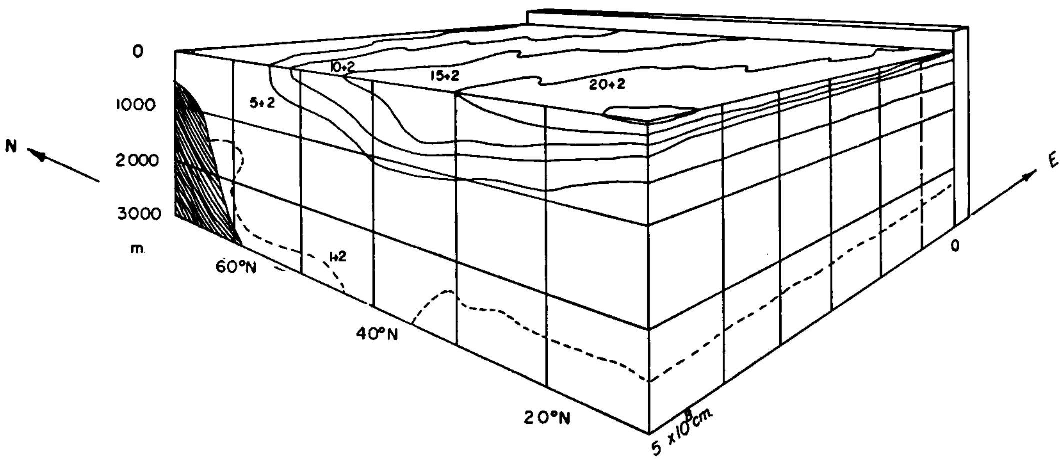

Fig. 2. Schematic temperature field observed in the North Atlantic Ocean.

The data for the North Atlantic Ocean is sketched roughly in figure 2. This diagram is based upon (i) the longitudinal profile of temperature on the western side of the Atlantic presented in Wüst's (1936) monograph on the oceanic stratosphere, (ii) Böhnnecke's (1936) chart of the North Atlantic surface temperature for the month of March, and (iii) the 1957 CRAWFORD section at 8° North latitude. The deep temperature of the Atlantic is 2° C, not 0° C, so in comparing isotherms, we write the 17° C isotherm in figure 2 as 15 + 2, and compare it with the 15° isotherm in figure 1. Certain similarities of the two figures are, of course, forced, for example we have chosen the surface temperature distribution in figure 1 to correspond roughly with that of figure 2. Also the vertical scale of figure 1 was drawn to look like that of figure 2, but actually matching determines the parameter $\varkappa$. Even allowing for the forced elements of similarity in the picture, the remaining portions correspond surprisingly well. Of particular interest is the fact that the isothermal surfaces are deepest in mid-latitudes—a fact which hitherto has been attributed to the surface convergence of wind-driven surface layers. Here of course, in figure 1, there is no wind acting; whereas, in the real ocean diagram, figure 2, we must be on the lookout for the effect of the wind which actually does act on the North Atlantic.

Before attempting to account for the effect of the wind, first we calculate various quantities such as the parameter $\varkappa$ and the vertical component of velocity by fitting the two figures 1 and 2 as they stand. Thus the real Atlantic Ocean is roughly 5,000 km wide and 10,000 km long, and taking $L_x = 5{,}000$ km, $L_y = 10{,}000$ km makes the horizontal scales of the two figures agree. Setting $= 10^{-4}\,\mathrm{sec}^{-1}$, $T_0 = 20°$, $T_1 = -2.5 \cdot 10^{-8}\,°\mathrm{C/cm}$, $\beta = 2 \cdot 10^{-13}\,\mathrm{cm}^{-1}\,\mathrm{sec}^{-1}$, $\alpha = 2 \times 10^{-4}/°\mathrm{C}$, $g = 10^3\,\mathrm{cm/sec}$. To fit the vertical scales we want the value Hz′ equal to 1 to correspond to the depth 1,000 meters at $y' = 0$ and $x' = -1$, and we obtain the value of the parameter $\varkappa$:

$$\varkappa = 10\ \mathrm{cm}^2/\mathrm{sec}$$

The vertical velocity at the bottom of the main thermocline may be written in terms of the thermocline thickness z_t defined as the depth at which the temperature is e^{-1} of the surface temperature locally. This useful relation is obtained by elimination of $\varkappa / \left(T_0 + \frac{T_1 f}{\beta} \right)$ between equations (25) and (26).

$$w(\infty) = (T_0 + T_1 y) \frac{\alpha \beta g}{3 f^2(y) x} z_t^2$$

If the thermocline thickness were 1,000 meters (as it appears at first glance to be at $y' = 0$, $x' = -1$) the sub-thermocline vertical velocity is

$$w(\infty) = 53 \times 10^{-5}\ \mathrm{cm\ sec}^{-1}$$

The expression for the vertical velocity in terms of thermocline thickness is interesting because it does not involve the parameter $\varkappa$ explicitly. It is very similar to the expression used by STOMMEL (1958) in an analysis of the abyssal circulation, but using the much more primitive model of VERONIS and STOMMEL (1957). We believe that both these values, of $\varkappa$ and $w(\infty)$ are an order of magnitude too high. The reason is that we are comparing a model in figure 1 with no winds, to a schematic representation of the real North Atlantic Ocean which does suffer being acted upon by winds. In mid-latitudes Montgomery (1936) has computed that there ought to be an convergent Ekman layer at the surface, directly beneath which there is a vertical component of velocity downwards, $w_e = -5 \times 10^{-5}\ \mathrm{cm\ sec}^{-1}$. We expect therefore that part of the warm water layer in the upper portions of figure 2 really must correspond to the wind-driven layer of thickness h, and that below this layer the proper thermohaline regime begins. The thickness of thermocline which we used before was too big. It included both h and z_t. A much better idea of the rate of exponential decay of temperature in the main thermocline $a x' = 1$, $y' = 0$ can be obtained from curve NA in figure 3.

Let us proceed by first writing the quantities L, h, and W_e in terms of the original coordinates. This is advisable because the dimensions of these quantities are rather odd

$$\frac{L}{2} = (\varepsilon / x)^{\frac{1}{2}} z_t$$

$$h = (\varepsilon / x)^{\frac{1}{2}} z_h$$

$$W_e = (x / \varepsilon)^{\frac{1}{2}} w_e$$

where z_t is the "thickness" of the thermocline, estimated by the shape of the temperature curve well below the inflection point at $\partial^2 \vartheta / \partial \zeta^2 = 0$ in the temperature curve, and z_h is the "depth" of the wind-driven layer which we identify in figure 3, for example, as the depth of the inflection point. The temperature at the inflection point is defined as ϑ_1. The information we obtain from the North Atlantic curve in figure 3 is as follows

$$z_t \cong -3.5 \times 10^8\ \mathrm{cm}$$

$$z_h \cong -9.0 \times 10^4\ \mathrm{cm}$$

$$\vartheta_1 \cong 10°\,\mathrm{C}$$

These estimates are for a geographical position corresponding to

$$x = -5 \times 10^8\ \mathrm{cm}$$
$$y = 0$$

at which

$$\varepsilon \cong 10^{-6}\ (°\mathrm{C\ sec})^{-1}$$

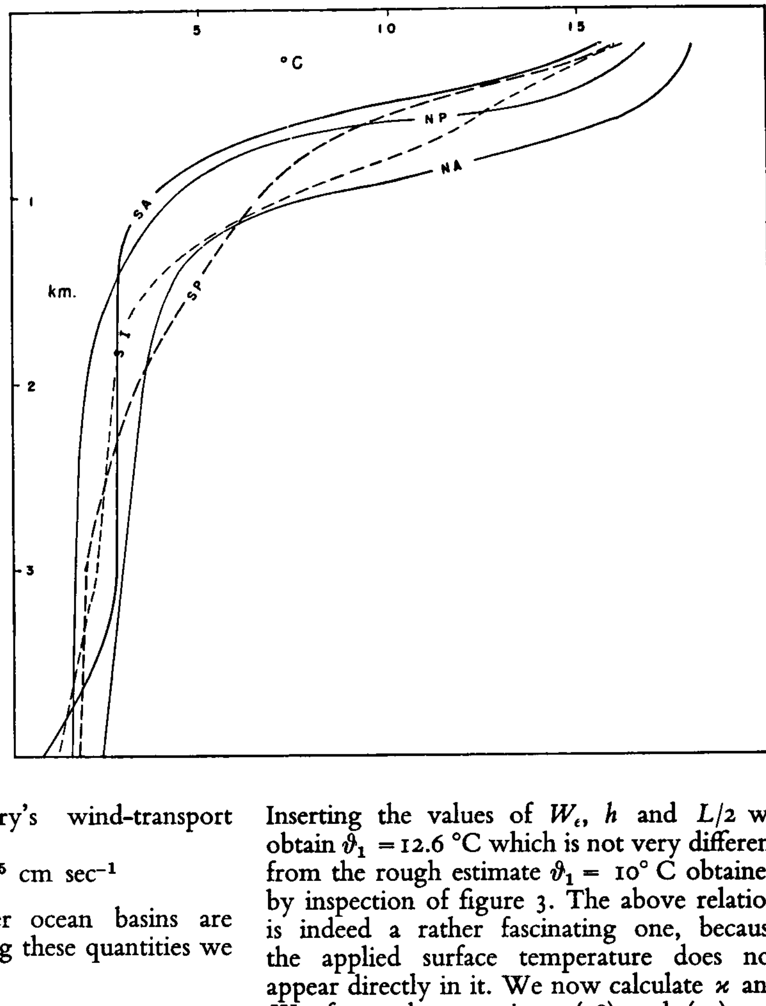

Fig. 3. Temperature vs Depth at mid-latitudes in western regions of various oceans. N = North, S = South, A = Atlantic, I = Indian, P = Pacific.

and from Montgomery's wind-transport charts

$$w_e \cong -5 \times 10^{-5} \text{ cm sec}^{-1}$$

The quantities for other ocean basins are somewhat different. Using these quantities we compute

$$\frac{L}{2} = 0.44 \text{ cm}^{2/3} (\text{sec °C})^{-1/3};$$

$$W_e = 4.0 \text{ cm}^{4/3} \text{ sec}^{-2/3} \text{°C}^{1/3}$$

$$h = 0.72 \text{ cm}^{2/3} (\text{sec °C})^{-1/3}; \quad \vartheta_1 = 10\text{° C}$$

According to the formulation in section 7, Case 2, these quantities are not all independent. We need only specify two, and the others are computable. However, there is a degree of uncertainty in identifying any one of these quantities from oceanographic data, so it is helpful to list them all, and then using the relations (36) and (37) inquiring whether they are mutually consistent. Thus, from equation (37) we have the remarkable relation

$$\vartheta_1 = \frac{W_e 2}{Lh}$$

Inserting the values of W_e, h and $L/2$ we obtain $\vartheta_1 = 12.6$ °C which is not very different from the rough estimate $\vartheta_1 = 10$° C obtained by inspection of figure 3. The above relation is indeed a rather fascinating one, because the applied surface temperature does not appear directly in it. We now calculate $\varkappa$ and W_∞ from the equations (38) and (35) and obtain

$$\varkappa = 0.85 \text{ cm}^2/\text{sec}$$

$$W_\infty = -2.5 \text{ cm}^{4/3} \text{ sec}^{-2/3} \text{ °C}^{1/3}$$

Thus the actual upwelling velocity beneath the thermocline obtained from by transformation to the original coordinates is

$$w_\infty = 3.1 \times 10^{-5} \text{ cm sec}^{-1}.$$

We believe that the order of magnitude of the mixing parameter $\varkappa$ and the deep upwelling velocity w_∞ computed from the North Atlantic temperature distribution in this manner is much more in accord with the estimates of the magnitude of mixing processes and of the deep-ocean transports, than the numbers obtained neglecting the wind. Thus in applying these ideas to the actual ocean it is essential to include the wind stress in the model. The two causes of oceanic circulation (wind-stress and thermohaline process) are connected in a basically non-linear fashion and cannot be super-posed as additive solutions. Fofonoff has already pointed this out for a zonal ocean. Of course, in this model, vertical integrals of the transports depend on wind-stress alone—as has now been known for twelve years.

The reader can now substitute the various constants into the expression for the temperature in the upper layer and see that it affords a reasonable approximation to the form of that actually observed.

The state of affairs in the North Atlantic Ocean is not markedly different from that in other oceans. The vertical distribution of temperature at 30° latitude near the western sides (but east of the western boundary currents) of other meridionally bounded oceanic basins is shown in figure 3. (NA, North Atlantic; SA, South Atlantic; S. I., South Indian; NP, North Pacific; SP, South Pacific.) Actually the thermocline in the North Atlantic is somewhat anomalously deep compared to those in other oceans, but the shape of the main thermoclines themselves are very similar: i.e. the value of z_t is about the same. The fact that the Pacific is twice as wide as the Atlantic is of no consequence because width of the ocean appears only as a cube root factor. Some of the complications in the temperature curves shown in figure 3 are due to salinity effects on the density field—but these are details which at the present stage of the theory do not merit immediate consideration.

The model shown in figure 1 exhibits a number of general features which ought to be mentioned. First, the warm water in mid ocean does not tend to move toward the poles—but flows with an equatorward meridional component. At low latitudes it turns west (since it cannot cross the equator, according to equation (6)) where presumably it forms a western boundary current when it encounters the western boundary of the ocean. The western boundary current is a higher-order dynamical regime which can occur under fairly general conditions of stratification, so there is no reason to doubt that the thermal circulation can be "closed" on the western side by such a boundary current—although we have not attempted a formal demonstration for this particular model. Everywhere in deep water there is a slow upward component of velocity. The way in which the solution for a single basin can be connected together with solutions for other basins to form a scheme descriptive of the general circulation of the world ocean has already been pointed out by STOMMEL (1958).

9. Comments on the parametric treatment of mixing, the quantity $\varkappa$

The inclusion of an eddy-conductivity, $\varkappa$, is a parametric way of including small scale vertical mixing processes into the theoretical model. Although this is a very common thing to encounter in the earlier oceanographical literature, it is a technique which present-day oceanographers are very reluctant to employ—except as a last resort. It means that we simply do not know anything about the physics of the mixing process. Despite our ignorance of the cause of mixing or the mechanism, we do have some idea, as we will show below, of the order of magnitude of the mixing. We take $\varkappa$, by these other considerations, to be of order of magnitude unity (c.g.s.), or five hundred times the molecular diffusion coefficient. This assumption has important consequences on how the model is to be set up, as was shown in equation (14). If $\varkappa$ were much greater than unity, equation (14) would become $\vartheta''=0$, yielding a linear form for ϑ which would have no thermocline-like character. If $\varkappa$ were much less than unity equation (14) would become $W\vartheta' - \gamma W'\vartheta_\eta = 0$, which describes a purely advective process. This equation, together with (15), has been solved by the method outlined in the beginning of section 4. The solution exhibits something like a thermocline, but its dependence on latitude does not agree with observations. Therefore, the choice for $\varkappa$ of order unity seems necessary. In the results we note that only the cube root of $\varkappa$ appears in the depth L, so that $\varkappa$ may vary by an order of magnitude without changing the results significantly.

In the calculations we have taken $\varkappa$ to be a constant. A more valid $\varkappa$ would probably in general be a function of depth. Were this the case, the first term of the heat equation (16) would be replaced by $[\varkappa(\xi)\vartheta'(\xi)]'$. In the resulting calculation, the first term of (21) would be replaced by $\varkappa(0)\vartheta'(0)$, and $\varkappa(0)$ would replace $\varkappa$ in all the results. This means that within the validity of the above calculations the form of $\varkappa$ is not critical. If more moments of equations (16) were taken, integral properties of $\varkappa(\xi)$ would enter.

The mean distribution of many oceanic variables like salinity, oxygen, carbon-14, etc., must depend to a large extent on the same mean velocity field and parametric mixing as those which determine the mean thermal field. For the most part these properties have no important dynamical effect—except, of course, salinity which can play a dominant role, but in subtropical areas usually plays a definitely secondary role in determining the density distribution. Therefore these properties are in a sense tracers whose distribution depends upon the thermal circulation and the parametric mixing, but which does not in turn effect them. Thus they provide a means for testing our model of the oceanic circulation for internal consistency. In order to make such a test it does not seem likely that very small scale experiments, or transitory phenomena will be very convincing because we are so completely ignorant about the nature of the mixing process that we do not even know its scale or spectrum, or how uniformly it acts in time. Under the circumstances we prefer to employ tracer phenomena whose horizontal and vertical and time scales are all comparable to those of the oceanic thermocline itself; and we want these tracers to be located quite close to or within the thermocline. We offer two examples from the Atlantic. In both cases salinity is the tracer.

Case 1: The first case is the layer of Antarctic Intermediate Water in the South Atlantic Ocean. At Meteor Profile VII in the eastern half of the ocean, between Stations 177 and 180, the salinity minimum lies at a depth of 700 meters, where the temperature is about 5° C, thus lying immediately beneath the main thermocline. The thickness of the layer is between 500 meters and 1,000 meters—depending upon how it is defined. The salinity in the minimum here is about 0.2‰ lower than that 200 meters above and below the minimum ($\partial^2 s/\partial z^2 \cong 10^{-9}$ ‰ cm^{-2}). The salinity gradient in the east direction is small ($\partial s/\partial x \cong 0$) as compared to that in the northward direction ($\partial s/\partial y \cong 7 \times 10^{-10}$ ‰ cm^{-1}). WÜST (1956) has made dynamical calculations of the velocity between stations 177—180 at 700 meters and finds a mean northward velocity of as little as $v = 0.2$ cm sec^{-1}. The mixing parameter computed from this situation is 0.14 cm^2 sec^{-1} [$\varkappa \cong (v\partial s/\partial x)/(\partial^2 s/\partial z^2)$]. According to our theory there ought to be an upward velocity of about 2×10^{-4} cm sec^{-1} at this level, so that we might expect the level of the minimum to slope upward toward the north with a slope of about 10^{-4}, which indeed it does.

Case 2: The second case to be discussed is a layer of maximum salinity which covers much of the southern North Atlantic in the area of the North Equatorial Current, and passes westward through the entire Caribbean, during which passage the intensity of the maximum decreases by about 0.3 ‰ ($\partial s/\partial x \cong +0.20 \times 10^{-8}$ ‰ cm^{-1}). Upon entering the Caribbean this layer is at a level of 125 meters and upon leaving through the Yucatan Channel it is 200 meters deep. The mean westward velocity of this layer of water is about 3 cm sec^{-1} ($u = -3.0$ cm sec.$^{-1}$) and its salinity maximum is about 0.6 ‰ greater than the salinity 100 meters above or below it ($\partial^2 s/\partial z^2 = 1.2 \times 10^{-8}$‰ cm^{-2}); thus one computes $\varkappa = 0.5$ cm^2 sec^{-1}, again a value more or less consistent with the requirements of our model. This saline layer originates at the surface in the subtropical surface salinity maximum, and sinks downward as it flows southward and westward. The layer is at a temperature of 20° C, hence it lies near to the top of the thermocline—and the vertical velocity should be downward here: this latter is in accord with the downward slope of the layer as it traverses the Caribbean (slope of 5×10^{-5}).

10. The Abyssal Circulation

As the reader can judge, this model of the thermohaline circulation implies many definite quantitative things about the circulation of the deep waters of the ocean. For example, we should be able to derive the total amplitude of the deep circulation of the ocean by integrating the deep upwelling velocity over the whole water-covered globe (in spherical coordinates instead of the beta-plane). In this way we should be able to predict the average age of the deep waters, a topic of current interest to geochemists, and a subject of much present day experimental enquiry, and sea-going activity. This further extension of the ideas here presented is treated in a separate paper by A. B. Arons and Henry Stommel which is being prepared for publication after this one.

REFERENCES

DEFANT, A., 1936: Entstehung und Erhaltung der tropo-sphärischen Sprungschicht. *Zeit. f. Geoph.* **12**, 7/8 pp. 281—286.

FOFONOFF, N., 1954: A steady flow in a frictionless homogeneous ocean. *J. Marine Res.* **13**, pp. 254—262.

GOLDSTEIN, S. (Editor), 1938: *Modern Developments in Fluid Dynamics*. Oxford University Press, p. 133 ff.

HIDAKA, K., 1950: Drift currents in an enclosed ocean. *Tokyo Univ., Geophys. Notes* **3** (23), pp. 1—23.

LEWIS, J. A. and CARRIER, G. F., 1949: Some remarks on the flat plate boundary layer. *Quart. Applied Math.* **7** p. 228.

LINEYKIN, P. S., 1955: On the determination of the thickness of the baroclinic layer in the sea. *Dokl. Akad. Nauk USSR* **101**, pp. 461—464.

— 1957: On the dynamics of the baroclinic layer in the ocean. *Dokl. Akad. Nauk USSR* **117**, pp. 971—974.

MUNK, W. H., and ANDERSON, E. R., 1948: Notes on a theory of the thermocline. *J. Marine Res.* **7**, pp. 276—295.

MUNK, W. H., 1950: On the wind-driven ocean circulation. *J. Met.* **7**, pp. 79—93.

STOMMEL, H., 1956: On the determination of the depth of no meridional motion. *Deep-Sea Res.* **3**, pp. 273—278.

— 1957: A survey of ocean current theory. *Deep-Sea Res.* **4**, pp. 149—184.

— 1958: *The Gulf Stream*. Univ. Cal. Press and Cambr. Univ. Press.

— 1958: The Abyssal Circulation. *Deep-Sea Res.* **5**, pp. 80—82.

VERONIS, G., 1956: Partition of energy between geostrophic and non-geostrophic oceanic motions. *Deep-Sea Res.* **3**, pp. 157—177.

VERONIS, G., and STOMMEL, H., 1956: The action of variable wind stresses on a stratified ocean. *J. Marine Res.* **15**, pp. 43—75.