On the Wind-Driven Ocean Circulation[1]

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Abstract

Some aspects of existing theories of the wind-driven ocean circulation are examined with particular emphasis on the question of the need for the inclusion of lateral eddy viscosity to provide a mechanism for balancing the applied wind torque. A new model is proposed according to which the ocean is divided into a southern and a northern portion, attention being restricted to the former which is itself subdivided into an interior region and a boundary region adjacent to the western shore. The equations of motion in terms of spherical coordinates are formally integrated over depth for both a homogeneous and a two-layer ocean. Approximate equations analogous to those used in existing theories are proposed for the interior region. Conditions in the boundary region are considered in an effort to determine the relative importance of the various terms in the equations. Based on these considerations approximate equations are derived for the boundary region. These imply the predominance of the pressure terms, the nonlinear inertia terms and the terms arising from the variation of the Coriolis parameter with latitude.

The approximate equations are transformed to surface coordinates and are applied to the homogeneous ocean and a two-layer ocean subjected to a simple wind distribution, yielding reasonable results. It is shown that the variation of the Coriolis parameter plays a fundamental role in the formation of the stream on the western shore. Simple physical interpretations of the results are presented including an explanation of the facts that no similar stream can be formed on the eastern shore and that the variation of depth in a two-layer ocean, when the Coriolis parameter is assumed constant, cannot give rise to an intense stream. Appropriate curves illustrating the dependence of the solutions on certain dimensionless parameters are given. When applied to the North Atlantic the theory gives reasonable results for the Gulf Stream north to, say, Cape Hatteras.

1. Introduction

Existing theories of the wind-driven circulation in closed ocean basins (SVERDRUP, 1947; MUNK, 1950) are based on the assumption that the principal features of the flow pattern are the result of the balance between wind, Coriolis, lateral friction and pressure forces and that all other contributions to the dynamic equations are, for purposes of an approximate investigation, negligible. This assumption leads to a linear problem, an approximate solution of which is obtained most conveniently by applying boundary layer analysis to the vorticity equation (MUNK and CARRIER, 1950). Although the circulation which is predicted is, at least qualitatively, very reasonable, the theory has been criticized especially on the grounds that the value of the coefficient of lateral eddy viscosity must be chosen empirically, and further, that the value which gives the proper width to the intense current on the western shore is considerably larger than that indicated by other, independent considerations.

In the following, other aspects of the abovementioned theories are considered and attention is devoted to the question of the importance of the lateral eddy viscosity. It has previously been argued that this must be included in every steady-state theory because it provides the forces of friction on the sides of the basin which give rise to the torque that

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must be present to balance the total torque exerted on the ocean by the wind. This problem is discussed in Section 2 where an equation for the balance of total moment of momentum about the center of mass of the ocean is derived and it is seen that this consideration alone is not sufficient to determine whether friction is required in the moment balance.

In Section 3 the equation of vorticity in terms of spherical coordinates is formally integrated over the depth of a layer of water which is considered to have uniform density (either the total ocean depth, if one deals with a homogeneous model, or the depth of the upper layer in a two-layer model). Approximate equations which apply to the interior region of the ocean and stipulate the predominance of wind, Coriolis, and pressure forces are derived in Section 4. In Section 5 the conditions in the boundary region in which an intense current is expected are discussed and we return to the problem of the need for including lateral eddy viscosity. Based on this discussion a model is proposed according to which the ocean is divided into a southern and a northern region. Attention is confined to the former which is again divided into an interior region, to which the equations derived in Section 4 apply, and a boundary region, adjacent to the western shore, for which approximate equations are derived by means of boundary layer analysis. These imply the predominance of the pressure terms, the non-linear inertia terms, and the terms arising from the variation of the Coriolis parameter with latitude, friction being neglected.[1]

The approximate equations are transformed to surface coordinates in Section 6 and are applied to a homogeneous ocean in Section 7. The theory predicts an intense current flowing northward along the western shore. The reason for the existence of this current from a physical point of view is discussed and it is shown why no similar current can exist on the eastern shore.

In Section 8 the theory is applied to the upper layer of a two-layer ocean. The role of a certain parameter involving the depth of the layer, the magnitude of the westward volume transport into the boundary region, the south-north dimension of the basin, the reduced gravity constant, and the variation of the Coriolis parameter with latitude, is studied. A proof is given that the variation of the Coriolis parameter is essential to the formation of an intense current even in a two-layer system. Numerical solutions are presented and it is concluded that, for reasonable values of the physical quantities involved, the barotropic and baroclinic models are not basically different as far as the formation of the stream is concerned. When applied to the North Atlantic the theory predicts the correct order of magnitude for the width of the Gulf Stream.

2. Moment of Momentum Balance

It has frequently been asserted (e.g. MONTGOMERY, 1940) that, for steady-state conditions, friction must be present on the bounding surfaces of the basin to provide the torque required to balance the total torque applied by the wind. Since it has been held reasonable to assume that the friction on the bottom is negligible, this argument has led to the conclusion that lateral eddy viscosity must be included in the equations of motion in order to give rise to a shear stress on the sides of the basin. In view of the obvious importance of this question, it will be examined in the following. The physical quantity of interest is the total moment of momentum about the center of mass, the momentum referring to the velocity recorder by an observer who is fixed to the earth. For steady-state conditions this moment must be constant.

The equation of motion is

$$\varrho \frac{Dq}{Dt} + \varrho 2 \boldsymbol{\Omega} \times \boldsymbol{q} + \varrho \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = = -\nabla p + \varrho \nabla \chi' + \boldsymbol{F} \qquad (2.1)$$

where q is the velocity of a particle as seen by an observer fixed to the earth, D/Dt is the material derivative, ϱ is the density (not necessarily uniform), $\boldsymbol{\Omega}$ is the earth's angular

[1] The probability that viscosity is unimportant in the Gulf Stream was suggested by H. Stommel in a discussion following formal papers delivered at the June 1954 Convocation at Woods Hole (the complete proceedings of which are to be published as a supplement to the Journal of Marine Research, Vol. 14, No. 4, Dec. 31, 1955), and in a privately printed pamphlet entitled "Why do our ideas about the ocean circulation have such a peculiarly dream-like quality?", April 1954.

rotation vector, $\boldsymbol{r}$ is the position vector of the particle with respect to the center of the earth and with respect to a coordinate system which rotates with the earth, p is the pressure, χ' is the gravitational potential, and $\boldsymbol{F}$ is the net force on a unit volume due to eddy viscosity and can be written $\nabla \cdot \sigma$ where σ denotes the stress dyad minus the contribution of the pressure p.

The equation of conservation of mass is

$$\frac{\partial \varrho}{\partial t} + \operatorname{div} \varrho \boldsymbol{q} = 0. \qquad (2.2)$$

Set

$$\nabla \chi' - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \nabla \chi' - \nabla \left(\frac{1}{2} \Omega^2 \tilde{\omega}^2 \right) \equiv \nabla \chi \qquad (2.3)$$

where $\tilde{\omega}$ is the perpendicular distance of a point from the axis of rotation of the earth, Ω is the absolute value of $\boldsymbol{\Omega}$ and χ is the apparent gravitational potential. Carrying out the vector multiplication of the resulting equation with $\boldsymbol{r}_c$, the position vector of a point with respect to the center of mass, and integrating over the total volume of the ocean, one has

$$\int_V \varrho \boldsymbol{r}_c \times \frac{D\boldsymbol{q}}{Dt} dV + \int_V \varrho \boldsymbol{r}_c \times (2\boldsymbol{\Omega} \times \boldsymbol{q})\, dV = - \int_V \boldsymbol{r}_c \times \nabla p dV + \int_V \varrho \boldsymbol{r}_c \times \nabla \chi dV + \int_V \boldsymbol{r}_c \times \nabla \sigma \cdot dV \qquad (2.4)$$

This equation can be transformed by making use of equation (2.2), the relations

$$\frac{D\boldsymbol{r}}{Dt} = \boldsymbol{q},$$

$$\int_V \varrho \frac{D\varphi}{Dt} dV = \frac{D}{Dt} \int_V \varrho \varphi dV, \qquad (2.5)$$

φ being any arbitrary continuous function with continuous first derivatives, and the transformation formulae relating volume and surface integrals, to give

$$\frac{D}{Dt} \int_V \boldsymbol{r}_c + \varrho \boldsymbol{q} dV + \int_V \boldsymbol{r}_c \times \varrho\, (2\, \boldsymbol{\Omega} \times \boldsymbol{q}) dV = \int_V \varrho \boldsymbol{r}_c \times \nabla \chi dV - \int_S \boldsymbol{r}_c \times \boldsymbol{n} p dS + \int_S \boldsymbol{r}_c \times \boldsymbol{T} dS \quad (2.6)$$

where S is the surface enclosing the volume V, $\boldsymbol{n}$ is the unit outer normal vector on S, and $\boldsymbol{T} \equiv \boldsymbol{n} \cdot \sigma$ and denotes that portion of the surface traction which is due to the frictional mechanism. It should be noted that equation (2.6) holds irrespective of the particular nature of the eddy viscosity (or molecular viscosity), the only additional requirement for its derivation being the symmetry of the stress dyad.

The left-hand side of equation (2.6) represents the rate of change of moment of momentum and consists of the rate of change as seen by an observer fixed to the earth and the contribution by the Coriolis forces. The right-hand side represents the moment of all external forces, the gravitational attraction, the pressure on the surfaces and the forces arising from viscosity. The last integral may be split into two portions, one involving integration over the free surface and thus representing the moment due to the wind, the other involving integration over the ocean bottom and sides. The contribution of the pressures to the moment is principally due to the fact that the free surface is disturbed by the motion and that the moments due to the pressures on the sides will then not, in general, add to zero. If one deals with a circular basin having a vertical shore and uniform depth, then the moment due to the pressures acting on the shore will be very small because the center of mass will be very close to the geometrical center.

For a steady-state solution the first term on the left of equation (2.6) must vanish. The contribution of the body force to the moment is negligible. Thus the moment balance involves the contributions of the Coriolis force, the pressure, the wind, and the friction on the shores and bottom, and it is seen that no immediate conclusion can be drawn from this consideration alone concerning the need for including friction on the sides, even if friction on the bottom is neglected from the start. The moment created by the wind may possibly be balanced by the Coriolis and pressure moments, or by the former alone if the basin is circular.

The preceding considerations in no way prove that lateral friction is not required; they only demonstrate that the question cannot be decided on the basis of moment balance alone. The problem will be discussed from a different point of view in Section 5.

3. Equations of Motion in Spherical Coordinates

Since an analytical investigation of the three-dimensional problem is prohibitively complicated, we follow the usual procedure of confining our attention to the integrals over depth of the velocity components and of the vorticity.

Using the spherical coordinates (r, ϑ, φ), the origin being at the center of the earth and ϑ being the colatitude, and corresponding velocity components q_1, q_2, q_3, the momentum equations are

$$\frac{\partial q_1}{\partial t} + \text{non linear terms} - 2\, q_3 \Omega \sin\vartheta = -\frac{1}{\varrho}\frac{\partial p}{\partial r} - g + F_1 \tag{3.1}$$

$$\frac{\partial q_2}{\partial t} + q_1 \frac{\partial q_2}{\partial r} + \frac{q_2}{r}\frac{\partial q_2}{\partial \vartheta} + \frac{q_3}{r\sin\vartheta}\frac{\partial q_2}{\partial \varphi} + \frac{q_1 q_2}{r} - \frac{q_3^2 \cot\vartheta}{r} - 2\, q_3 \Omega \cos\vartheta = -\frac{1}{\varrho}\frac{1}{r}\frac{\partial p}{\partial \vartheta} + F_2 \tag{3.2}$$

$$\frac{\partial q_3}{\partial t} + q_1 \frac{\partial q_3}{\partial r} + \frac{q_2}{r}\frac{\partial q_3}{\partial \vartheta} + \frac{q_3}{r\sin\vartheta}\frac{\partial q_3}{\partial \varphi} + \frac{q_1 q_3}{r} + \frac{q_2 q_3}{r}\cot\vartheta + 2 q_2 \Omega \cos\vartheta + 2 q_1 \Omega \sin\vartheta = \frac{1}{\varrho}\frac{1}{r\sin\vartheta}\frac{\partial p}{\partial \varphi} + F_3 \tag{3.3}$$

Here (F_1, F_2, F_3) denote the contributions of the eddy viscosity, g denotes the gravitational acceleration, and the centrifugal force terms which give rise to the ellipticity of the undisturbed free surface of the ocean have been neglected.

We shall make either of two assumptions concerning the distribution of density; a) we consider the "homogeneous" ocean which has uniform density everywhere and is subsequently assumed to be of uniform depth when undisturbed; b) we consider a "two-layer" ocean which consists of two superposed layers each of uniform density. In the latter case we shall be concerned only with the motion in the upper, lighter layer, and the equations of motion refer to that layer alone. Hence, in both cases, the equation expressing the conservation of mass is that for an incompressible fluid:

$$\frac{\partial}{\partial r}(r^2 \sin\vartheta\, q_1) + \frac{\partial}{\partial \vartheta}(r \sin\vartheta\, q_2) + \frac{\partial}{\partial \varphi}(r q_3) = 0 \tag{3.4}$$

The radial component of vorticity is

$$\zeta_1 = (\text{curl}\, \boldsymbol{q})_1 = \frac{1}{r^2 \sin\vartheta}\left[\frac{\partial}{\partial \vartheta}(r\sin\vartheta\, q_3) - \frac{\partial}{\partial \varphi}(r q_2)\right] \tag{3.5}$$

The first component equation of the vorticity equation is obtained by performing the operation appearing on the right side of (3.5) on equations (3.3) and (3.2) in the place of q_3 and q_2, respectively, and making use of equation (3.4):

$$\begin{aligned}
&\frac{\partial \zeta_1}{\partial t} + q_1 \frac{\partial \zeta_1}{\partial r} + \frac{q_2}{r}\frac{\partial \zeta_1}{\partial \vartheta} + \frac{q_3}{r\sin\vartheta}\frac{\partial \zeta_1}{\partial \varphi} + \\
&+ \frac{1}{r}\frac{\partial q_1}{\partial \vartheta}\frac{\partial q_3}{\partial r} + \frac{1}{r^2}\frac{\partial q_1}{\partial \vartheta} q_3 - \frac{1}{r\sin\vartheta}\frac{\partial q_1}{\partial \varphi}\frac{\partial q_2}{\partial r} - \\
&- \frac{1}{r^2\sin\vartheta}\frac{\partial q_1}{\partial \varphi} q_2 + \frac{2}{r} q_1 \zeta_1 - \\
&- \frac{\zeta_1}{r^2 \sin\vartheta}\frac{\partial}{\partial r}(r^2\sin\vartheta\, q_1) + \frac{4}{r}\Omega\cos\vartheta\, q_1 - \\
&- \frac{2}{r^2}\Omega\cot\vartheta \frac{\partial}{\partial r}(r^2 \sin\vartheta\, q_1) + \\
&+ \frac{2}{r}\Omega\sin\vartheta \frac{\partial q_1}{\partial \vartheta} - \frac{2}{r}\Omega \sin\vartheta\, q_2 = \\
&= \frac{1}{r^2\sin\vartheta}\left[\frac{\partial}{\partial \vartheta}(r\sin\vartheta\, F_3) - \frac{\partial}{\partial \varphi}(r F_2)\right]
\end{aligned} \tag{3.6}$$

The rate of change of the first component of vorticity as we follow a particle is

$$\frac{D\zeta_1}{Dt} = \frac{\partial \zeta_1}{\partial t} + q_1 \frac{\partial \zeta_1}{\partial r} + \frac{q_2}{r}\frac{\partial \zeta_1}{\partial \vartheta} + \frac{q_3}{r\sin\vartheta}\frac{\partial \zeta_1}{\partial \varphi} \tag{3.7}$$

Note that this is not the same as the first component of $D\boldsymbol{\zeta}/Dt$, $\boldsymbol{\zeta}$ being the vorticity vector. We shall find it convenient to focus our attention on that portion of the rate of change of ζ_1 which is associated with the particle's horizontal motion only and therefore define

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{q_2}{r}\frac{\partial}{\partial \vartheta} + \frac{q_3}{r\sin\vartheta}\frac{\partial}{\partial \varphi} \tag{3.8}$$

We now integrate equation (3.6) over the depth of the total ocean in case a) and over the upper layer in case b). In both cases the limits can be written $r = h_b(\vartheta, \varphi, t)$ on the bottom and $r = h_s(\vartheta, \varphi, t)$ on the free surface with h_b being constant in case a). Letting

$$f = 2\Omega \cos \vartheta \tag{3.9}$$

and making use of (3.8) one obtains, after some manipulation

$$\int_{h_b}^{h_s} \frac{d\zeta_1}{dt} dr - (q_1\zeta_1)\Big|_{h_b}^{h_s} + 2\int_{h_b}^{h_s} q_1 \frac{\partial \zeta_1}{\partial r} dr +$$
$$+ \int_{h_b}^{h_s} \left(\frac{1}{r}\frac{\partial q_1}{\partial \vartheta}\frac{\partial q_3}{\partial r} + \frac{1}{r^2}\frac{\partial q_1}{\partial \vartheta} q_3 - \frac{1}{r \sin \vartheta}\frac{\partial q_1}{\partial \varphi}\frac{\partial q_2}{\partial r} - \right.$$
$$\left. - \frac{1}{r^2 \sin \vartheta}\frac{\partial q_1}{\partial \varphi} q_2 \right) dr + \int_{h_b}^{h_s} \frac{df}{dt} dr - (q_1 f)\Big|_{h_b}^{h_s} +$$
$$+ 2\Omega \sin \vartheta \int_{h_b}^{h_s} \frac{1}{r}\frac{\partial q_1}{\partial \vartheta} dr =$$
$$= \frac{1}{\sin \vartheta} \int_{h_b}^{h_s} \frac{1}{r^2}\left[\frac{\partial}{\partial \vartheta}(r \sin \vartheta F_3) - \frac{\partial}{\partial \varphi}(rF_2)\right] dr \tag{3.10}$$

where the symbol $(\ )\Big|_{h_b}^{h_s}$ stands for $(\ )_{r=h_s} - (\ )_{r=h_b}$.

We now evaluate the frictional terms in (3.10). Let

$$F_i = \frac{1}{\varrho}\frac{1}{r^2}\frac{\partial}{\partial r}\left(A_r r^2 \frac{\partial q_i}{\partial r}\right) + F_i^*, \quad i = 2, 3 \tag{3.11}$$

where A_r denotes the radial eddy viscosity and F_2^*, F_3^* are used for the sake of brevity to represent the contributions due to the coefficient of lateral eddy viscosity A_L the magnitude of which is so far arbitrary. Assuming that A_r is independent of ϑ and φ and neglecting the variation of r within the range of the integration, one obtains for the right side of equation (3.10)

$$\frac{1}{\varrho}\frac{\cot \vartheta}{r}\left(A_r \frac{\partial q_3}{\partial r}\right)\Big|_{h_b}^{h_s} + \frac{1}{\varrho}\frac{1}{r}\left(A_r \frac{\partial^2 q_3}{\partial r \partial \vartheta}\right)\Big|_{h_b}^{h_s} -$$
$$- \frac{1}{\varrho}\frac{1}{r \sin \vartheta}\left(A_r \frac{\partial^2 q_2}{\partial r \partial \varphi}\right)\Big|_{h_b}^{h_s} +$$
$$+ \frac{1}{\sin \vartheta}\int_{h_b}^{h_s} \frac{1}{r^2}\left[\frac{\partial}{\partial \vartheta}(r \sin \vartheta F_3^*) - \frac{\partial}{\partial \varphi}(rF_2^*)\right] dr \tag{3.12}$$

The pertinent components of shear stress are

$$\tau_{r\vartheta} = A_r\left(\frac{1}{r}\frac{\partial q_1}{\partial \vartheta} - \frac{q_2}{r} + \frac{\partial q_2}{\partial r}\right)$$
$$\tau_{r\varphi} = A_r\left(\frac{1}{r \sin \vartheta}\frac{\partial q_1}{\partial \varphi} - \frac{q_3}{r} + \frac{\partial q_3}{\partial r}\right) \tag{3.13}$$

Since the depth of the ocean is much smaller than the radius of the earth, only the last term in each of the brackets in equation (3.13) is important. Expression (3.12) can then be written[1]

$$\frac{1}{\varrho}\left\{\frac{1}{r}\frac{\partial}{\partial \vartheta}\left(\tau_{r\varphi}\Big|_{h_b}^{h_s}\right) + \frac{\cot \vartheta}{r}\left(\tau_{r\varphi}\Big|_{h_b}^{h_s}\right) - \right.$$
$$- \frac{1}{r \sin \vartheta}\frac{\partial}{\partial \varphi}\left(\tau_{r\vartheta}\Big|_{h_b}^{h_s}\right) -$$
$$- \frac{1}{r}\left(\left[\frac{\partial}{\partial r}\left(A_r \frac{\partial q_3}{\partial r}\right)\right]\frac{\partial h}{\partial \vartheta}\right)\Big|_{h_b}^{h_s} +$$
$$\left. + \frac{1}{r \sin \vartheta}\left(\left[\frac{\partial}{\partial r}\left(A_r \frac{\partial q_2}{\partial r}\right)\right]\frac{\partial h}{\partial \varphi}\right)\Big|_{h_b}^{h_s}\right\} +$$
$$+ \frac{1}{\sin \vartheta}\int_{h_b}^{h_s} \frac{1}{r^2}\left[\frac{\partial}{\partial \vartheta}(r \sin \vartheta F_3^*) - \frac{\partial}{\partial \varphi}(rF_2^*)\right] dr \tag{3.14}$$

where $\left(\left[\ \right]\frac{\partial h}{\partial \vartheta}\right)\Big|_{h_b}^{h_s}$ is an abbriviation for

$\left[\ \right]_{r=h_s}\frac{\partial h_s}{\partial \vartheta} - \left[\ \right]_{r=h_b}\frac{\partial h_b}{\partial \vartheta}$ and $\frac{\partial}{\partial \vartheta}\left(\tau_{r\varphi}\Big|_{h_b}^{h_s}\right)$ $= \frac{\partial}{\partial \vartheta}\left[\tau_{r\varphi}(h_s, \vartheta, \varphi, t) - \tau_{r\varphi}(h_b, \vartheta, \varphi, t)\right]$.

[1] For details see: G. W. MORGAN; On the integration over depth of the equations for the wind-driven ocean circulation; Reference No. 54–89; unpublished manuscript of the Woods Hole Oceanographic Institution, December 1954.

Note that the three terms containing the shear stresses are approximately $\frac{1}{\varrho}(\text{curl }\tau)_1\Big|_{h_b}^{h_s}$, where τ is the shear stress vector on the surfaces h_s or h_b. The additional surface terms arise from the fact that

$$\left(\frac{\partial}{\partial\vartheta}\frac{\partial q_3}{\partial r}\right)\Bigg|_{h_b}^{h_s}+\frac{\partial}{\partial\vartheta}\left[\left(\frac{\partial q_3}{\partial r}\right)\Bigg|_{h_b}^{h}\right]$$

4. Interior Equations

In this section we derive approximate equations applicable to the interior of the ocean, i.e. to a region sufficiently far removed from all shores. A more precise definition of the "interior region" will be given in Section 5. We assume, as do other investigators, that all terms which are nonlinear in the velocity components, as well as the contributions of lateral eddy viscosity are negligibly small there. It is further assumed that the pressure is hydrostatic everywhere in the ocean, i.e. that all terms involving velocity components in (3.1) are negligible. These assumptions can be readily justified by examining the orders of magnitude of the terms in the equations. Equation (3.1) becomes

$$\frac{\partial p}{\partial r}=\varrho g \tag{4.1}$$

Integration of equation (4.1) together with the boundary condition $p=0$ at $r=h_s$ gives

$$p=\varrho g\,(h_s-r). \tag{4.2}$$

For the two-layer model, we assume that all velocities are negligible in the lower layer so that the pressure gradients vanish there. This leads to the two relations

$$\frac{\partial h_b}{\partial\vartheta}=-\frac{\varrho}{\Delta\varrho}\frac{\partial h_s}{\partial\vartheta},\ \frac{\partial h_b}{\partial\varphi}=-\frac{\varrho}{\Delta\varrho}\frac{\partial h_b}{\partial\varphi} \tag{4.3}$$

where $\Delta\varrho$ is the density difference between the two layers. Letting

$$D=h_s-h_b \tag{4.4}$$

we can write

$$\frac{\partial h_s}{\partial\vartheta}=\frac{\partial D}{\partial\vartheta},\ \frac{\partial h_s}{\partial\varphi}=\frac{\partial D}{\partial\varphi} \tag{4.5}$$

for the homogeneous ocean (h_b having been assumed constant) and

$$\frac{\partial h_s}{\partial\vartheta}=\frac{\Delta\varrho}{\varrho}\frac{\partial D}{\partial\vartheta},\ \frac{\partial h_s}{\partial\varphi}=\frac{\Delta\varrho}{\varrho}\frac{\partial D}{\partial\varphi} \tag{4.6}$$

for the two-layer model, where it has been assumed that $\Delta\varrho\ll\varrho$.

The momentum equations (3.2) and (3.3) become, using (3.11) and (4.2)

$$-2\Omega q_3\cos\vartheta=-g\frac{1}{r}\frac{\partial h_s}{\partial\vartheta}+ +\frac{1}{\varrho}\frac{1}{r^2}\frac{\partial}{\partial r}\left(A_r r^2\frac{\partial q_2}{\partial r}\right) \tag{4.7}$$

$$2\Omega q_2\cos\vartheta+2\Omega q_1\sin\vartheta=-g\frac{1}{r\sin\vartheta}\frac{\partial h_s}{\partial\varphi}+ +\frac{1}{\varrho}\frac{1}{r^2}\frac{\partial}{\partial r}\left(A_r r^2\frac{\partial q_3}{\partial r}\right) \tag{4.8}$$

Before writing the approximate form of the vorticity equation (3.10) it is convenient to evaluate the term $(q_1 f)\Big|_{h_b}^{h_s}$. Applying the kinetic boundary conditions

$$\frac{D}{Dt}(h_s-r)=\frac{D}{Dt}(h_b-r)=0 \tag{4.9}$$

we have

$$(q_1 f)\Big|_{h_b}^{h_s}=f\left(\frac{q_2}{r}\frac{\partial h}{\partial\vartheta}+\frac{q_3}{r\sin\vartheta}\frac{\partial h}{\partial\varphi}\right)\Bigg|_{h_b}^{h_s} \tag{4.10}$$

Evaluating the right side by means of equations (4.7) and (4.8), we have

$$(q_1 f)\Big|_{h_b}^{h_s}=-\left(\frac{2\Omega}{r}\sin\vartheta q_1\frac{\partial h}{\partial\vartheta}\right)\Bigg|_{h_b}^{h}+ +\frac{1}{\varrho}\left\{\left(\frac{1}{r^3}\frac{\partial}{\partial r}\left[A_r r^2\frac{\partial q_3}{\partial r}\right]\frac{\partial h}{\partial\vartheta}\right)\Bigg|_{h_b}^{h_s}- -\left(\frac{1}{r^3\sin\vartheta}\frac{\partial}{\partial r}\left[A_r r^2\frac{\partial q_2}{\partial r}\right]\frac{\partial h}{\partial\varphi}\right)\Bigg|_{h_b}^{h_s}\right\} \tag{4.11}$$

When this expression is substituted into equation (3.10) the last two terms cancel the corresponding terms on the right side of equation (3.10) as given by expression (3.14), provided we neglect the variation of r in equation (4.11).

The resulting vorticity equation (without nonlinear and lateral eddy viscosity terms) may be simplified further by noting that the slopes of the top and bottom surfaces will be much less than one, so that $q_1 \ll q_2$. Thus, only the term in df/dt is important on the left side and, using (3.8), equation (3.10) becomes approximately

$$-\frac{2\Omega}{r}\sin\vartheta\int_{h_b}^{h_s} q_2 dr = \frac{1}{\varrho}\left\{\frac{1}{r}\frac{\partial}{\partial\vartheta}\left(\tau_{r\varphi}\Big|_{h_b}^{h_s}\right) + \frac{\cot\vartheta}{r}\left(\tau_{r\varphi}\Big|_{h_b}^{h_s}\right) - \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\left(\tau_{r\vartheta}\Big|_{h_b}^{h_s}\right)\right\} \quad (4.12)$$

Integration of the approximate momentum equations (4.7) and (4.8) leads to

$$-2\Omega\cos\vartheta\int_{h_b}^{h_s} q_3 dr = -g'\frac{1}{r}D\frac{\partial D}{\partial\vartheta} + \frac{1}{\varrho}(\tau_{r\vartheta})\Big|_{h_b}^{h_s} \quad (4.13)$$

$$2\Omega\cos\vartheta\int_{h_b}^{h_s} q_2 dr = -g'\frac{1}{r\sin\vartheta}D\frac{\partial D}{\partial\varphi} + \frac{1}{\varrho}(\tau_{r\varphi})\Big|_{h_b}^{h_s} \quad (4.14)$$

where we have used (4.4) to (4.6) and have neglected q_1, and

$$\left.\begin{aligned} g' &\equiv g && \text{in the homogeneous ocean} \\ g' &\equiv \frac{\Delta\varrho}{\varrho}g && \text{in the two-layer ocean} \end{aligned}\right\} \quad (4.15)$$

The integrated continuity equation is readily shown to be

$$\frac{\partial}{\partial\vartheta}\int_{h_b}^{h_s}\sin\vartheta q_2 dr + \frac{\partial}{\partial\varphi}\int_{h_b}^{h_s} q_3 dr = 0 \quad (4.16)$$

This relation is approximate in that the variation of r over the range of integration has been neglected. Introducing the "transport" components Q_2, Q_3, equation (4.16) is satisfied by defining the transport stream function ψ by the relations

$$\begin{aligned} Q_2 &\equiv \int_{h_b}^{h_s} q_2 dr \equiv -\frac{1}{r\sin\vartheta}\frac{\partial\psi}{\partial\varphi} \\ Q_3 &\equiv \int_{h_b}^{h_s} q_3 dr \equiv -\frac{1}{r}\frac{\partial\psi}{\partial\vartheta} \end{aligned} \quad (4.17)$$

Equations (4.12) to (4.14) and (4.17) are the equations which we shall apply to the interior of the ocean.

5. Boundary Equations

a) Preliminary discussion.

To conform with reality, the circulation to be predicted by theory will have to exhibit relatively large velocities and velocity gradients at least near the western shore. Hence the approximations made in Section 4 for the interior region of the ocean cannot be expected to hold there. This is otherwise evident from a mathematical point of view since the approximate equations (4.12) to (4.16) do not constitute a system of a sufficiently high order to yield solutions which satisfy all boundary conditions. Hence these equations are not adequate for the entire enclosed ocean.

It has been assumed by other investigators (e.g. MUNK, 1950) that the influence of lateral eddy viscosity, while negligible in the interior, becomes important near the shore. The theory based on this assumption has led to a circulation pattern which looks remarkably realistic. As was already mentioned in the introduction, the theory has been criticized, however, because of the need for the empirical choice of the value of the coefficient of lateral eddy viscosity and the fact that the value which gives the proper width to the stream on the western shore is larger than one might expect from independent considerations. It is useful to consider the merits of this "viscous" theory from other points of view.

It was stated in the introduction that the torque due to the frictional forces on the shores has, at times, been held to be the torque which balances the torque applied by the wind. Now the viscous theory shows that the magnitude of the shore friction torque depends on the value of the viscosity (the shear stress on the shore being proportional to the viscosity and the normal derivative of the tangential velocity component, and the latter varying as viscosity to the minus two-thirds power). Hence, if the argument were correct, the value of the viscosity should be chosen so as to satisfy the torque balance! Fortunately the discussion in Section 2 relieves us of this requirement. If the friction torque is essential to the torque balance, it is certainly not the only contribution.

Consider now briefly, from a physical point of view, the role of the shore in producing an intense current. The solution of the interior vorticity and continuity equations (4.12) and (4.16) with a realistic wind distribution and neglecting bottom friction (see Section 6) leads to a southward transport component over the southern interior. The equations do not determine the direction of the other component. This evidently depends on the flow near the coast. Whichever it is, in a model which requires that the circulation be confined essentially to the northern hemisphere, the stipulation of an eastern or western shore must give rise to an intense northward current on one of the coasts, since the water which flows toward this coast has to be turned to the north to satisfy continuity. The stream will have to be narrow and hence intense if it is not to interfere with the concept of the interior region. Thus, the boundary condition of vanishing normal velocity on the shore is responsible for turning the current northward; the mechanism which accounts for the narrowness of this current (and which will determine on which of the two shores the current exists) remains to be investigated, but it appears reasonable that the usual viscous boundary condition of no slip will not have to play an essential role in the creation of the stream. This is also indicated by the viscous theory which shows that even if the extreme condition of zero normal derivative of the tangential transport is specified on the coast, the width and intensity of the stream are essentially unaffected. If, however, the no-slip condition is not essential, then the function of viscosity may not be paramount in the formation of the stream. It is only necessary to have a system of equations which is of sufficiently high order to be capable of yielding solutions which can satisfy the condition of vanishing normal transport. This, of course, can be accomplished by including the nonlinear inertia terms. The inclusion of viscosity, on the other hand, leads to a higher order system which requires another and, it seems, extraneous boundary condition.

It is important to note that the preceding discussion, having been restricted to the formative stage of the northward stream, does not necessarily suggest that lateral viscosity will not be important anywhere in the system. Since our study of the moment of momentum balance gave no conclusive answer to this question, it may be useful to examine it here from another point of view.

Imagine that the flow is contained between two solid concentric spheres and that it is created by the application of body forces uniformly distributed over the depth, rather than by surface forces. If there is no top or bottom friction we may expect the motion to be independent of r and to have vanishing radial velocity. The vorticity equation (3.6) then becomes

$$\frac{d}{dt}(\zeta_1+f) = (\mathrm{curl}\,\boldsymbol{B})_1 + \begin{matrix}\text{lateral eddy}\\ \text{viscosity terms}\end{matrix} \qquad (5.1)$$

where $\boldsymbol{B}$ represents the body forces. If $(\mathrm{curl}\,\boldsymbol{B})_1$ is everywhere positive, (as we may expect $(\mathrm{curl}\,\boldsymbol{\tau})_1\big|^{h_1}$, the curl of the wind stress, to be), and if viscosity is absent, then equation (5.1) says that the quantity ζ_1+f will always increase as we follow a particle. If the system is closed, this leads to a nonsteady flow. Hence, for steady flow, lateral friction must be important at least in some region through which every particle will have to pass.

This conclusion also follows immediately from energy considerations. The direction of the flow to be expected in the interior is such that the body force would do positive work on the system. In the absence of lateral eddy viscosity, however, there is no mechanism for dissipating the resulting increase in kinetic energy.

The author has so far not succeeded in carrying out an analogous examination of the variation of ζ_1+f for the actual problem. It may be noted, however, that in our formulation the transfer of the wind stress to the water occurs by means of vertical eddy viscosity and that, therefore, a mechanism for energy dissipation exists even in the absence of lateral eddy viscosity. Thus, the energy argument breaks down.

b) A new model.

The considerations presented so far suggest that we adopt a new model for our system. Let us imagine the ocean between, say, 10° latitude and 50° latitude to consist of a southern and a northern portion with the dividing circle of latitude at about 35°. The southern portion is further divided as shown. The figure also shows a typical streamline of the anticipated circulation, most, or perhaps all, of the streamlines being expected to pass through all three regions.

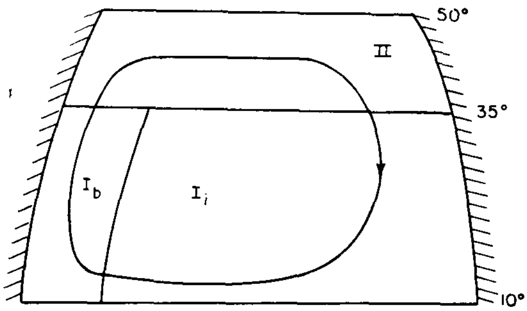

Fig. 1. The three regions of a new ocean model. I_i: interior region; I_b: frictionless stream region; II: northern region; nonsteady and lateral friction effects possibly important.

Our theory will apply to the interior region I_i and the boundary region I_b; (it will be seen that no boundary region is required on the eastern shore). In I_i the equations of Section 4 are applicable. In I_b nonlinear inertia, but not viscosity, will be included. In the Atlantic, I_b is to include the Gulf Stream north to about Cape Hatteras. In region II nonlinear, viscous and nonsteady effects may be important. This region includes the Gulf Stream meanders. If it turns out that lateral viscosity must be present somewhere in the system, its influence will be confined to region II. It is quite possible that the analysis of the problem in regions I and II together may have to include instability of the meanders, i.e. nonsteady effects. These, too, are assumed to be present in II only. The analysis of the flow in I is based on the assumption that this flow can be studied without inquiring into the conditions in II. Thus, whatever occurs in II, the flow which emerges from there into I_i is such that it obeys our interior equations. Similarly, the flow entering II from I_b is supposed to be determined by conditions in I_i and I_b alone.

c) Order of magnitude estimates.

The terms in the vorticity equation (3.10) may be grouped into three categories: (i) terms nonlinear in the velocities, (ii) terms due to the earth's rotation, (iii) terms resulting from eddy viscosity. The relative importance of each group will be examined in the following subsection by the usual procedures of boundary analysis. In the present section we carry out a preliminary simplification by comparing the relative magnitudes of the terms within each group.

(i) It will be assumed that the shores are given by two meridians. We must then expect that, in the boundary region, $|q_3| \ll |q_2|$ and derivatives parallel to the shore, $\left(\frac{\partial}{\partial \vartheta}\right)$, will be much smaller than those normal to the shore, $\left(\frac{\partial}{\partial \varphi}\right)$. Hence, from equation (3.5), $\zeta_1 \approx -(1/r \sin \vartheta)(\partial q_2/\partial \varphi)$. The magnitude of q_1 is readily estimated from the kinetic boundary condition, equation (4.9), in terms of q_2, q_3 and the slopes of h_s and h_b. For the homogeneous ocean q_1 is very small due to the small surface slope. For the two-layer ocean the principal contribution is due to the slope of h_b, the thermocline, the variation of h_b being of the order of the depth D itself.

Examining the third integral in equation (3.10), it is clear that the first, second and fourth terms are negligible compared with the third.

The contributions of $\int_{h_b}^{h_s} (d\zeta_1/dt) dr$ and $(q_1\zeta_1)\Big|_{h_b}^{h_s}$ are seen to be of equal importance in the two-layer case. To compare these terms with the remaining nonlinear terms involving radial differentiation, we must obtain an estimate of $\partial q_2/\partial r$. The second, third and fourth terms of equation (3.10) have the following orders of magnitude for a two-layer model:

$$[(q_1/r)(\partial q_2/\partial \varphi)]_{r=h_b}; \quad (q_1/r)_{r=h_b}\left[(\partial q_2/\partial \varphi)\Big|_{h_b}^{h_s}\right];$$

$$(\partial q_1/\partial \varphi)_{r=h_b}\left[(q_2/r)\Big|_{h_b}^{h_s}\right].$$

(For a homogeneous ocean q_1 is so small everywhere that all these terms are negligible.) The second and third of the above expressions involve a difference in the values at h_s and h_b of the quantities $\partial q_2/\partial\varphi$ and q_2, respectively. If these differences are of the order of the quantities themselves, then all terms must be expected to be of equal importance. We assume in the following that these differences are much smaller than the quantities themselves and hence that the term $(q_1 \zeta_1)\Big|_{h_b}^{h_s}$, predominates. This implies that the top and bottom shear stresses which constitute the principal cause of the radial variation of q_2 are relatively unimportant in the boundary region, thus leaving q_2 essentially uniform with depth. This condition can probably be relaxed somewhat because our estimates of the third and fourth terms are likely to be on the high side, q_1 and $\partial q_1/\partial\varphi$ having been taken out of the integrals and evaluated at $r = h_b$ where they will be greatest. Nevertheless it may be important to note that the justification for omitting the terms under consideration is not beyond question.

(ii) As in (i), the first two terms involving Ω in equation (3.10) are seen to be of the same order, while the third one is negligible.

(iii) We turn to expression (3.12) for the terms due to eddy viscosity. Of the three terms involving radial eddy viscosity the last one will predominate. The principal contribution due to lateral eddy viscosity comes from $F_2{}^*$ and is approximately

$$\frac{1}{\varrho r^3 \sin^3\vartheta}\int_{h_b}^{h_s} A_L \frac{\partial^3 q_2}{\partial\varphi^3}\, dr$$

A_L denoting the lateral eddy viscosity.

The simplified vorticity equation (3.10) now becomes

$$\int_{h_b}^{h} \frac{d}{dt}(\zeta_1 + f)\, dr - [q_1(\zeta_1 + f)]\Big|_{h_b}^{h_s} = $$
$$= -\frac{1}{\varrho}\frac{1}{r}\left(A_r \frac{\partial^2 q_2}{\partial r \partial\varphi}\right)\Big|_{h_c}^{h_s} -$$
$$- \frac{1}{r^3 \sin^3\vartheta}\int_{h_b}^{h_s} A_L \partial^2 q_2/\partial\varphi^2 \cdot dr \qquad (5.2)$$

with $\zeta_1 \cong -(1/r \sin\vartheta)(\partial q_2/\partial\varphi)$.

d) Boundary layer analysis.

In this subsection the technique of boundary layer analysis is employed to examine the relative magnitudes of the terms in (5.2). This technique assumes that a boundary layer type of solution exists. If it does, then the procedure readily yields an estimate of some of the properties of the solution; in this case, of the width and intensity of the stream.

We first assume that the essence of the phenomenon of interest to us will not be lost if we neglect the radial variation of the velocity components and of A_L within the boundary layer for the purpose of evaluating the integrals in equation (5.2). We also assume that the radial eddy viscosity term is not important. This assumption may not be valid for a homogeneous or a two-layer ocean, or perhaps even for a more realistic model, but an argument in its favor might be as follows. At the top surface the radial velocity gradient will be zero in the absence of a meridional wind. At the bottom surface the radial eddy viscosity may be sufficiently small to keep $(A_r\, \partial^2 q_2/\partial r \partial\varphi)_{r=h_b}$ small. We make the assumption in spite of the uncertainty of its validity because of the desirability of investigating the boundary solution in the complete absence of friction. Equation (5.2) becomes

$$D\frac{d(\zeta_1 + f)}{dt} - [q_1(\zeta_1 + f)]\Big|_{h_b}^{h_s} =$$
$$= -\frac{D}{\varrho r^3 \sin^3\vartheta} A_L \frac{\partial^2 q_2}{\partial\varphi^2} \qquad (5.3)$$

or, using the kinetic boundary conditions (4.9),

$$\frac{d}{dt}\left(\frac{\zeta_1 + f}{D}\right) = -\frac{1}{D\varrho r^3 \sin^3\vartheta} A_L \frac{\partial^3 q_2}{\partial\varphi^3} \quad (5.4)$$

Equation (5.4) is rendered dimensionless by the following transformations

$$\left.\begin{array}{ll} q_2 = q^* q_2' & q_3 = q^* q_3' \\ \frac{1}{r}\frac{\partial}{\partial\vartheta} \approx \frac{1}{s}\frac{\partial}{\partial\vartheta'} & \frac{1}{r}\frac{\partial}{\partial\varphi} \approx \frac{1}{s}\frac{\partial}{\partial\varphi'} \\ \beta = \frac{2\,\Omega}{r} & \end{array}\right\} \quad (5.5)$$

where q^* is a characteristic velocity taken to be a measure of the zonal velocity in the transition region from the interior to the boundary layer, s is the south-north extent of region I, and $\sin\vartheta$ is approximated by one. We further restrict ourselves for the moment to an homogeneous ocean or to a two-layer model with such a deep upper layer that the depth may be regarded as uniform. Equation (5.4) then becomes approximately

$$\frac{q^*}{s^2\beta}\left(q_2'\frac{\partial^2 q_2'}{\partial\vartheta'\partial\varphi'} + q_3'\frac{\partial^2 q_2'}{\partial\varphi'^2} - \cos\vartheta\, q_2'\frac{\partial q_2'}{\partial\varphi'}\right) + q_2' = \frac{A_L}{s^3\beta}\frac{\partial^3 q_2'}{\partial\varphi'^3} \qquad (5.6)$$

According to our discussion in Sections 5 a) and 5 b), we expect that the nonlinear terms will be important in the boundary layer. Since they are negligible in the interior, the parameter

$$N \equiv \frac{q^*}{s^2\beta} \qquad (5.7)$$

must be much less than one. Hence the terms cannot be important in the boundary layer unless the derivatives, and possibly the functions themselves, are large there. We therefore set

$$\varphi' = N^m\varphi'', \quad q_2' = N^{-n}q_2'' \qquad (5.8)$$

where m, n are to be determined and are expected to be positive. We hope to find a transformation (5.8) such that q_2'' $(\vartheta_1'\varphi'')$, q_3' (ϑ', φ'') and their derivatives are of order one in the boundary layer. The magnitude of each term will then be indicated by its coefficient.

The continuity equation (4.16) shows that $m = n$ and equation (5.6) becomes

$$N^{1-3m}\left(q_2''\frac{\partial^2 q_2''}{\partial\vartheta'\partial\varphi''} + q_3'\frac{\partial^2 q_2''}{\partial\varphi''^2} - \cos\vartheta\, q_2''\frac{\partial q_2''}{\partial\varphi''}\right) + N^{-m}q_2'' = N^{-4m}\frac{A_L}{\varrho s^3\beta}\frac{\partial^3 q_2''}{\partial\varphi''^3} \qquad (5.9)$$

Since we expect nonlinear and Coriolis terms to balance, we must have $m = \frac{1}{2}$. The terms due to lateral eddy viscosity will be unimportant if $N^{-3/2}A_L/\varrho s^3\beta = \beta^{1/2}A_L/\varrho q^{*3/2} \ll 1$. The width of the stream will be of order $sN^{1/2} = (q^*/\beta)^{1/2}$, and the meridional velocity component will be of order $q^*N^{-1/2} = s(q^*\beta)^{1/2}$. With $\beta = 2\times 10^{-13}\cdot(\text{cm sec})^{-1}$, $q^x = 10$ cm sec^{-1} this gives approximately 70 km for the width of the stream, which is the correct order of magnitude. The condition for smallness of the friction terms places an upper bound of approximately 5×10^6 cm^2 sec^{-1} on the magnitude of A_L. Assuming that this corresponds to actual conditions and that, therefore, lateral eddy viscosity is negligible, the vorticity equation (5.4) becomes

$$\frac{d}{dt}\left(\frac{\zeta_1 + f}{D}\right) = 0 \qquad (5.10)$$

Since, in this equation, the variation of ζ_1, q_2, q_3 with depth has, effectively, been neglected, the derivative d/dt may be interpreted as the rate of change as one follows a vertical column of water in its horizontal motion, and the equation (5.10) states that the potential vorticity associated with such a column is conserved during its motion in the boundary region. Using equation (4.17), equation (5.10) can be integrated along a "transport line", i.e. a line of constant ψ, to give

$$\frac{\zeta_1 + f}{D} = F(\psi) \qquad (5.11)$$

where F is a function which must be determined by matching the distribution of $(\zeta_1 + f)/D$ in the boundary with the distribution in the interior near the edge of the boundary layer.

e) The momentum equations in the boundary layer.

The considerations contained in the preceding sections may now be applied to the integrals over depth of the momentum equations (3.2) and (3.3) to derive approximate equations which are consistent with the assumptions made in deriving the vorticity equation. They are

$$\int_{h_b}^{h_s}\left(\frac{q_2}{r}\frac{\partial q_2}{\partial\vartheta} + \frac{q_3}{r\sin\vartheta}\frac{\partial q_2}{\partial\varphi}\right)dr - 2\Omega\cos\vartheta\int_{h_b}^{h_s} q_3\,dr = -g'\frac{1}{r}D\frac{\partial D}{\partial\vartheta} \qquad (5.12)$$

$$2\Omega\cos\vartheta\int_{h_b}^{h} q_2\,dr = -g'\frac{1}{r\sin\vartheta}D\frac{\partial D}{\partial\varphi} \qquad (5.13)$$

We note that the meridional transport component is geostrophic, but that the zonal component is not, the nonlinear terms being as important as the Coriolis contribution.

Neglecting the variation of the integrands with depth, we derive an approximate Bernoulli equation by multiplying equation (5.12) by q_2, equation (5.13) by q_3, adding the resulting equations to obtain

$$\frac{d}{dt}\left(\frac{q_2^2}{2}\right)+g'\frac{dD}{dt}=0, \qquad (5.14)$$

and finally integrating along a streamline (or transport line), giving

$$\frac{q_2^2}{2}+g'D=B(\psi) \qquad (5.15)$$

where B is an arbitrary function which must be determined by matching the boundary with the interior solution.

6. Transformation to Surface Coordinates

Having derived the equations in terms of spherical coordinates in an effort not to lose any important effects of the spherical ocean shape by premature introduction of plane coordinates, we are now ready to make this transformation to give the equations a more familiar appearance.

Our boundary layer analysis has been based on the assumption that the derivative $\partial/\partial\varphi$ represents differentiation normal to the coast. Hence two meridians are taken to represent the coast lines. The simplest coordinate system will then be one in which these are coordinate lines. Hence, set

$$\left.\begin{array}{lll} x=R\varphi, & y=R(K-\vartheta) & z=r-R \\ u=q_3, & v=-q_2 & \\ U=Q_3, & V=-Q_2 & \\ \tau_x=\tau_{r\varphi}, & \tau_y=-\tau_{r\vartheta} & \end{array}\right\} \qquad (6.1)$$

where K is the colatitude of the southern boundary of region I, R is the radius of the undisturbed ocean surface. Note that while y measures distance along a meridian of a sphere of radius R, x does not measure exact distance along a circle of latitude. We have

$$\frac{\partial}{\partial r}=\frac{\partial}{\partial z};\ \frac{\partial}{\partial\vartheta}=-R\frac{\partial}{\partial y};\ \frac{\partial}{\partial\varphi}=R\frac{\partial}{\partial x} \qquad (6.2)$$

If, instead of x, an alternative coordinate x' which does measure distance along a circle of latitude were chosen, (e.g. $x'=\varphi R\sin\vartheta$), then the most convenient geometrical shape in the x'—y plane, the rectangle, would represent a less realistic configuration than does the rectangle in the x—y plane which corresponds to an ocean whose east and west shores are meridians. Moreover, the velocity component v would then be an inconvenient variable since it would not represent the velocity in the direction of the constant x' curves.[1]

The interior equations (4.12), (4.13), (4.14), (4.17) become

$$\beta\sin\vartheta\int_{h_b-R}^{h_s-R} v\,dz=$$

$$=\frac{1}{\varrho}\left(-\frac{\partial\tau_x}{\partial y}+\frac{\cot\vartheta}{R}\tau_x+\frac{1}{\sin\vartheta}\frac{\partial\tau_y}{\partial x}\right)_{z=h_b-R} \qquad (6.3)$$

$$2\Omega\cos\vartheta\int_{h_b-R}^{h_s-R} u\,dz=-\frac{g'}{2}\frac{\partial D^2}{\partial y}+\frac{1}{\varrho}(\tau_y)_{z=h_s-R} \qquad (6.4)$$

$$-2\Omega\cos\vartheta\int_{h_b-R}^{h_s-R} v\,dz=-$$

$$-\frac{g'}{2\sin\vartheta}\frac{\partial D^2}{\partial x}+\frac{1}{\varrho}(\tau_x)_{z=h_s-R} \qquad (6.5)$$

$$V=\int_{h_b-R}^{h_s-R} v\,dz=\frac{1}{\sin\vartheta}\frac{\partial\psi}{\partial x},\ U=\int_{h_b-R}^{h_s-R} u\,dz=-\frac{\partial\psi}{\partial y} \qquad (6.6)$$

[1] The reader may wish to contrast our system with that of MUNK 1950 and MUNK-CARRIER 1950 whose x coordinate measures distance, with the result that the rectangle in the x—y plane does not correspond to a realistic ocean shape. This leads Munk-Carrier to consider a triangle. The rectangle in our x—y plane is perhaps an equally good approximation to the real ocean shape.

The agreement of corresponding terms in Munk's equations and in ours is not complete. This is due to the fact that Munk's equations have apparently not been derived by a systematic transformation of variables and coordinates from a spherical system; hence there is some question concerning the relation of Munk's variables to those in the actual spherical system.

where we have approximated $1/r$ by $1/R$ and, consistent with the assumption already made in the derivation of the boundary equations, have neglected the shear stress at the bottom surface $z=h_b-R$.

The boundary equations (5.11), (5.12), (5.13), (5.15) become

$$\frac{\frac{1}{\sin\vartheta}\frac{\partial v}{\partial x}+2\Omega\cos\vartheta}{D}=F(\psi) \qquad (6.7)$$

$$\int_{h_b-R}^{h_s-R}\left(v\frac{\partial v}{\partial y}+\frac{u}{\sin\vartheta}\frac{\partial v}{\partial x}\right)dz+2\Omega\cos\vartheta\int_{h_b-R}^{h_s-R}u\,dz=$$
$$=-\frac{g'}{2}\frac{\partial D^2}{\partial y} \qquad (6.8)$$

$$-2\Omega\cos\vartheta\int_{h_b-R}^{h_s-R}v\,dz=-\frac{g'}{2\sin\vartheta}\frac{\partial D^2}{\partial x} \qquad (6.9)$$

$$\frac{v^2}{2}+g'D=B(\psi) \qquad (6.10)$$

Just as in the derivation of the vorticity equation (6.7), the integrals appearing in equations (6.8) to (6.9) will be evaluated approximately by neglecting the z variation of the integrands.

7. The homogeneous Ocean with simple Wind Distribution

The homogeneous model is important principally because of the insight that will be gained into the role of density stratification in the formation of the western stream by comparing the behaviour of this model with that of a two-layer ocean. The most important question to be investigated in this connection is whether either the variation of the Coriolis parameter with latitude, or the density stratification, or both, are indispensable. Because of its simplicity, the homogeneous model also readily affords insight into the entire phenomenon of the coastal stream.

Assume the wind stresses

$$\tau_x=-W\left(1-\frac{y^2}{s^2}\right),\quad \tau_y=0,\quad 0\le y\le s \qquad (7.1)$$

s being the south-north extent of region I and W being a constant. Equation (6.3) becomes

$$\beta\sin\vartheta V_i=\frac{W}{\varrho}\left[-\frac{2y}{s^2}+\frac{\cot\vartheta}{R}\left(1-\frac{y^2}{s^2}\right)\right] \qquad (7.2)$$

where the subscript i is used to denote quantities in the interior region I_i. For $\vartheta\ge 55°$ and $K=75°$, $\cot\vartheta\le .7$ and $s\approx .35\ R$, so that the second term due to the curl of the wind is considerably smaller than the first term over most of the range. Accordingly, we neglect it. Using equation (6.6)

$$\frac{\partial\psi_i}{\partial x}=-\frac{2yW}{\varrho\beta s^2} \qquad (7.3)$$

$$\psi_i=-\frac{2yW}{\varrho\beta s^2}[x+l(y)] \qquad (7.4)$$

$$U_i=\frac{2W}{\varrho\beta s^2}(x+l)+\frac{2yW}{\varrho\beta s^2}\frac{dl}{dy} \qquad (7.5)$$

where $l(y)$ is an arbitrary function. Since we do not expect a strong current on the eastern coast, (the impossibility of such a current will shortly be demonstrated), we anticipate that the interior solution will be valid all the way to this coast. Hence, we want $U=0$ at $x=a$, $x=0$ and $x=a$ denoting the western and eastern shores. Thus in the region I_i

$$\psi_i=\frac{2yW}{\varrho\beta s^2}(a-x) \qquad (7.6)$$

$$U_i=-\frac{2W}{\varrho\beta s^2}(a-x) \qquad (7.7)$$

We now examine the boundary equations. From equations (6.6) and (6.7) and neglecting the radial variation of v,

$$\frac{\frac{1}{\sin^2\vartheta}\frac{\partial}{\partial x}\left(\frac{1}{D}\frac{\partial\psi}{\partial x}\right)+2\Omega\cos\vartheta}{D}=F(\psi). \qquad (7.8)$$

Since $g'=g$ in the homogeneous case, it is clear from equations (6.8) and (6.9) that the surface slopes will be very small and that, for a reasonable average depth, the depth will be essentially uniform. Expanding $\cos\vartheta$ in a power series about $\vartheta=K$ and retaining the first two terms

$$2\Omega\cos\vartheta\approx 2\Omega\cos K+2\Omega\frac{y}{R}\sin K \qquad (7.9)$$

or

$$f\approx f_K+\beta_K y,\quad \beta_K\equiv\frac{2\Omega}{R}\sin K,\quad f_K\equiv 2\Omega\cos K \qquad (7.10)$$

and equation (7.8) becomes

$$\frac{1}{\sin^2 \vartheta} \frac{1}{D} \frac{\partial^2 \psi}{\partial x^2} + f_K + \beta_K y = G(\psi) \quad (7.11)$$

where $G(\psi) = DF(\psi)$, D being regarded constant.

According to boundary layer analysis, as discussed in Section 5, the $\partial/\partial x$ derivatives are large only in the boundary layer and decrease as one approaches the interior. The transition from boundary to interior regions is, of course, continuous, but, to simplify the terminology, let us define some suitable distance $x = L$ as the "edge" of the boundary layer, L being large enough so that the boundary layer solution there is essentially equal to the interior solution in the same neighborhood. We then have

$$f_K + \beta_K y = G(\psi_i) \text{ at } x = L \quad (7.12)$$

But, from equation (7.6),

$$\psi_i(L, y) \approx \frac{2 Wa}{\varrho \beta s^2} y \equiv U^* y \quad (7.13)$$

since we expect $L \ll a$, and where U^* denotes the zonal transport into the boundary layer. Hence

$$f_K + \beta_K y = G(U^* y) \quad (7.14)$$

and

$$G(\psi) = f_K + \frac{\beta_K}{U^*} \psi \quad (7.15)$$

Substitution into equation (7.11) yields

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{\sin^2 \left(K - \frac{y}{R}\right) D \beta_K}{U^*} \psi = $$
$$= - \sin^2 \left(K - \frac{y}{R}\right) D \beta_K y \quad (7.16)$$

If we consider a closed ocean basin, the coast $x = 0$ and the southern boundary $y = 0$ form a continuous transport line so that $\psi(0, y) = 0$. This boundary condition together with the matching condition as the interior is approached (equation 7.13) determine the constants of integration. The solution is

$$\psi = U^* y \left[1 - e^{-\left(\frac{\sin^2 (K - y/R) D\beta_K}{U^*}\right)^{1/2} x}\right] \quad (7.17)$$

and, from equation (6.6)

$$V = (U^* D \beta_K)^{1/2} y e^{-\left(\frac{\sin^2 (K - y/R) D\beta_K}{U^*}\right)^{1/2} x} \quad (7.18)$$

Thus the theory predicts an intense northward stream on the western coast, the width of the stream being given by, say,

$$\sin \left(K - \frac{y}{R}\right) \cdot L \approx 4 \left(\frac{u^*}{\beta_K}\right)^{1/2}$$

and the intensity of the northward velocity being of order $s(u^* \beta_K)^{1/2}$, where $u^* = U^*/D$. These results are seen to agree with the orders of magnitude previously derived directly from boundary analysis considerations in Section 5 d). Expressions (7.17), (7.18) are simpler and more revealing if we set

$$x \sin \left(K - \frac{y}{R}\right) = X \quad (7.19)$$

X measuring true distance along a circle of latitude.

Curves 1 a of Figures 2 and 3 show the variation of ψ/Us and V/U^* with dimensionless distance X/s from the western shore at $y = s$ for values of U^* (10^5 cm^2 sec^{-1}) and s (2×10^8 cms) corresponding to the North Atlantic and for a depth D (or D^* in the terminology of Section 8) of 4×10^5 cms which corresponds to the total depth of the ocean. Hence the graphs may be interpreted as representing the Gulf Stream which would exist if the motion were barotropic. The stream is quite narrow (its width is about 40 km); the transport is very large near the coast and decreases very rapidly as the distance from the coast increases. The predicted flow is seen to exhibit what might be regarded as the principal qualitative property of the observed circulation—a westward interior transport being turned into a narrow, intense stream near the western coast—and the width of this stream has the appropriate order of magnitude.

The results obtained may be given a very simple physical interpretation. The vorticity equation (6.7), for D essentially constant, states that the absolute vorticity of a particle, $\zeta_1 + f$, is conserved. Since f increases as the particle travels northward, ζ_1 must decrease, and since ζ_1 $\left(\text{or } \frac{\partial v}{\partial X}\right)$ is approximately zero when the particle enters the boundary region it must become negative. Particles which approach the coast at low latitude, i.e. with small f, must undergo a large increase in f to reach a certain more northerly latitude, and hence acquire a large negative value of ζ_1 (or $\partial v/\partial X$). Those approaching with larger f need not suffer such a great change in f to reach the same northerly latitude and hence acquire less negative $\partial v/\partial X$. This gives rise to the exponential type of decline in current intensity from the coast to the interior.

It is now clear that the interior transport approaching the west coast could equally well be turned into an intense southward current if we did not stipulate that $y=0$ be a streamline in the boundary region, i.e. that the circulation be enclosed on the south. Following the same physical reasoning as before, a southward current means a decrease in f and hence positive $\partial v/\partial X$ to conserve vorticity. Further, the particles approaching the coast with larger f must acquire larger positive $\partial v/\partial X$ than must those approaching with smaller f. Hence we obtain a boundary layer solution with v having its greatest negative value at the coast and increasing to zero as the interior is approached.

Any flow pattern intermediate between the two already discussed, with a portion of the westward interior current being turned into an intense northward stream and the balance into an intense southward stream is also possible. The entire infinite family of solutions is obtained by stipulating the boundary condition

$$\psi(0, y) = \gamma s U^*, \quad 0 \le \gamma \le 1. \qquad (7.20)$$

This gives

$$\psi = U^*(\gamma s - y)\, e^{-\left(\frac{D\beta_K}{U^*}\right)^{1/2} X} + U^* y \qquad (7.21)$$

$$V = -(U^* D\beta_K)^{1/2}(\gamma s - y)\, e^{-\left(\frac{D\beta_K}{U^*}\right)^{1/2} X} \qquad (7.22)$$

Thus all particles approaching the boundary region north of $y=\gamma s$ (but, of course, south of $y=s$ to remain within region I) turn northward, those approaching between $y=0$ and $y=\gamma s$ turn southward, the transport line $\psi=U^*\gamma s$ dividing at $x=0$ to form the shoreline both south and north of $y=\gamma s$.

Which of all these solutions is the appropriate one depends on the appropriate choice of the boundary condition and this in turn is governed by considerations of mass conservation in the overall system.

Other solutions may be obtained, for example, by giving $\psi(0, y)$ a negative value. This gives a northward stream consisting not only of the water approaching the coast from the interior and being turned north, but also of water introduced into the stream in the boundary region across $y=0$, this extra flow preventing the transport line $\psi=0$ from reaching the coast.

Another important result which is evident immediately on physical grounds is that the factors which create an intense stream on the west coast could not create a similar stream on the east coast. To show this, let us suppose that the zonal transport component in the interior is eastward. If this transport is to turn northward, f must increase along a transport line, hence $\partial v/\partial X$ must become negative. Further, particles approaching with smaller f must acquire larger negative $\partial v/\partial X$, than those approaching with greater f. Hence the solution would require positive v, negative $\partial v/\partial X$, and v tending to zero as one leaves the boundary region and approaches the interior region. Obviously, these are incompatible conditions. A similar argument rules out a southward stream. Thus we see that a boundary layer type solution is possible in region I if the zonal transport is westward, but not if the zonal transport is eastward.

Mathematically, this is seen as follows. Since ψ_i is zero on $y=0$, and since $U_i = -\partial\psi_i/\partial y > 0$ for an eastward interior current, ψ_i would be negative for $y>0$; in particular ψ_i would be negative at the edge of the intense stream on the east coast. The matching with the boundary layer solution, analogous to equation (7.12), would then lead to an expression for $G(\psi)$ analogous to (7.15) with a minus sign in front of the term containing ψ. This gives rise to a plus sign for the ψ term in the equation analogous to (7.16) for the boundary layer stream function, and hence to trigonometric instead of exponential solutions. Hence no boundary layer type solution is possible.

To avoid confusion, it should be emphasized that the above conclusion does not in any way conflict with the well known problem of ideal, irrotational fluid theory, in which a flow impinges on a solid boundary placed normal to the undisturbed flow at infinity. This flow is not of the boundary layer type and the tangential current is just as wide and no more intense than the normal current. This type of flow was excluded from our investigations as soon as we made approximations appropriate to boundary layer type solutions.

It appears that the simple homogeneous ocean model contains the essential features of the type of circulation one expects. In the following section we investigate to what extent the two-layer model alters the phenomenon.

8. The Two-Layer Ocean with Simple Wind Distribution[1]

We consider the same problem as in Section 7 and make the same approximations except that now the variation of the depth D is taken into account. The interior relations (7.3) to (7.7) remain unchanged. We shall require a relation between U^x and D_i at the edge of the boundary layer. From equations (6.4) and (7.7), and setting $2\Omega\cos\vartheta = f \approx f_K + \beta_K y$, we have

$$D_i^2 = \left(f_K y + \frac{\beta_K y^2}{2}\right)\frac{4W}{\varrho\beta s^2 g'}(a-x) + C(x) \qquad (8.1\,\text{a})$$

where $C(x)$ is still arbitrary. If D^* is the depth at $y=0$ and at the edge of the boundary layer $x=L$, then $C(L)=D^{*2}$ and

$$D_i^2(L,y) = \frac{2U^x}{g'}\left(f_K y + \frac{\beta_K y^2}{2}\right) + D^{*2} \qquad (8.1\,\text{b})$$

Another expression for $D_i^2(x,y)$ may be derived by using the momentum equation (6.5). It is approximately

$$D_i^2 = -\left(f_K y + \frac{\beta_K y^2}{2}\right)\frac{4W}{\varrho\beta s^2 g'}x - \frac{2W\sin K}{\varrho g'}x + + C_1(y) \qquad (8.1\,\text{c})$$

Comparison with equation (8.1 a) yields

$$C(x) = D_a^2 + \frac{2W\sin K}{\varrho g'}(a-x) \qquad (8.2)$$

where D_a is a constant and denotes $D_i(a,y)$.

Instead of using the vorticity equation (6.7) it is more convenient, in the two-layer case, to use the momentum equation (6.9) and the Bernoulli equation (6.10). From equations (6.9) and (6.6)

$$\psi = \frac{g'}{2f}D^2 + C_3(y) \qquad (8.3)$$

where $C_3(y)$ is arbitrary and is determined by matching ψ and D with the interior solutions. We have from equations (7.13), (8.1) and (8.3)

$$U^* y = \frac{g'}{2f}\left[\frac{2U^*}{g'}\left(f_K y + \frac{\beta_K y^2}{2}\right) + D^{*2}\right] + C_3 \qquad (8.4)$$

whence

$$C_3(y) = \frac{\beta_K y^2 U^*}{2f} - \frac{g' D^{*2}}{2f} \qquad (8.5)$$

Substituting in equation (8.3) and solving for D^2 we obtain

$$D^2 = D^{*2} + \frac{2f}{g'}\psi - \frac{\beta_K y^2 U^*}{g'} \qquad (8.6)$$

Equation (8.6) provides one relation between the unknowns ψ and D. A second relation is obtained from the Bernoulli equation (6.10). The unknown function $B(\psi)$ is again determined by matching with the interior solution. As one leaves the boundary region, equation (6.10) becomes

$$g'\left[D^{*2} + \frac{2U^*}{g'}\left(f_K y + \frac{\beta_K y^2}{2}\right)\right]^{1/2} = B(U^* y) \qquad (8.7)$$

[1] Some weeks following the final preparation of this paper Dr. J. G. Charney (1955) published a theory of the Gulf Stream much like the one contained in the following section. Charney's aim is to study a two-layer model which is fashioned to fit as closely as possible the observed topography of the thermocline between the Florida Straits and Cape Hatteras and the observed volume transports in the Stream, in order to determine how closely the theory can reproduce the observed flow. By contrast, this author's primary goal has been to critically analyze some of the questions connected with the formulation of a suitable theory and the development of the pertinent equations, and to investigate the mechanism which gives rise to the intense coastal current. This is done with a view to clarifying such problems as the roles of density stratification and the variation of the Coriolis parameter, the possibility of the existence of a current on the eastern boundary, the direction of the intense current, etc. In order not to obscure any of these aspects of the problem the simple model used to study the homogeneous ocean is retained in this section, where our theory is applied to the two-layer system, and no attempt is made to fit the model more closely to the Gulf Stream.

giving

$$B(\psi) = g'\left[D^{*2} + \frac{2f_K}{g'}\psi + \frac{\beta_K}{g'U^*}\psi^2\right]^{1/2} \quad (8.8)$$

Substitution into equation (6.10) yields

$$\frac{v^2}{2} = g'\left\{\left[D^{*2} + \frac{2f_K}{g'}\psi + \frac{\beta_K}{g'U^*}\psi^2\right]^{1/2} - D\right\} \quad (8.9)$$

Finally, replacing v by $(1/D\sin\vartheta)\,\partial\psi/\partial x$ from equation (6.6) and eliminating D by using equation (8.6), the following equation for ψ is obtained:

$$\left(\frac{\partial\psi}{\partial x}\right)^2 = \\ = 2g'\sin^2\left(K - \frac{y}{R}\right)\left[D^{*2} + \frac{2f}{g'}\psi - \frac{\beta_K y^2 U^*}{g'}\right] \\ \left\{\left[D^{*2} + \frac{2f_K}{g'}\psi + \frac{\beta_K}{g'U^*}\psi^2\right]^{1/2} - \right. \\ \left. - \left[D^{*2} + \frac{2f}{g'}\psi - \frac{\beta_K y^2 U^*}{g'}\right]^{1/2}\right\} \quad (8.10)$$

The appropriate boundary condition is that $x=0$ be a streamline. As in Section 7 the particular value to be chosen for $\psi(o, y)$ depends on continuity considerations for the entire system. We shall restrict our numerical solutions to the model discussed in Section 7, i.e. to the case where the circulation is confined to a region north of $y=0$, so that

$$\psi(o, y) = 0 \quad (8.11)$$

We first note the very important result $\partial\psi/\partial x=0$ for all x if $\beta_K=0$. Thus, if the variation of the Coriolis parameter with latitude were neglected, no boundary stream could be produced. From a different point of view, the result shows that the variation in depth alone and its effect on the potential vorticity (equation (6.7)) cannot give rise to an intense stream. Thus, in the two-layer ocean, as in the homogeneous model, the variation of f is the essential phenomenon and it remains to investigate how the resulting flow pattern is modified by density stratification. More physical insight into this conclusion can perhaps be gained by the following consideration. Consider the change of relative vorticity ζ_1 $\left(\text{or } \frac{1}{\sin\vartheta}\frac{\partial v}{\partial x}\right)$ as one follows a streamline. Since, according to equation (6.7), potential vorticity is conserved, a change in ζ_1 will be due either to a change in f or to a change in D. We see from equation (8.6), however, that a change of D along a given streamline is due solely to a change in f. Hence, if f is constant, ζ_1 is constant along each streamline and since it is approximately zero near the edge of the boundary layer it must be zero everywhere. Thus the existence of β_K is the primary cause of the phenomenon. Inasmuch as β_K together with the small value of g' in the two-layer model give rise to a considerable change of depth along a streamline, and this change in turn effects the potential vorticity and hence the current, the latter is actually dependent on the depth (and hence on the density stratification) as well as on β_K.

Proceeding now with the analysis of the differential equation (8.10), we replace x by X according to equation (7.19) and then introduce the non-dimensional quantities $\bar{X}$, $\bar{y}$, $\bar{\psi}$ by the relations

$$X = s\bar{X},\quad y = s\bar{y},\quad \psi = U^*s\bar{\psi}$$

so that $\bar{\psi}$ approaches $\bar{y}$ as one approaches the interior. Equation (8.10) becomes

$$\left(\frac{\partial\bar{\psi}}{\partial\bar{X}}\right)^2 = \frac{2\beta_K^{3/2}s^3}{U^{*1/2}g'^{1/2}}[\varepsilon + \delta\bar{\psi} + 2\bar{y}\bar{\psi} - \bar{y}^2] \\ \{[\varepsilon + \delta\bar{\psi} + \bar{\psi}^2]^{1/2} - [\varepsilon + \delta\bar{\psi} + 2\bar{y}\bar{\psi} - \bar{y}^2]^{1/2}\} \quad (8.13)$$

where

$$\varepsilon = \frac{D^{*2}g'}{U^*\beta_K s^2} = \frac{D^*g'}{u^*\beta_K s^2},\quad \delta = \frac{2f_K}{\beta_K s} \quad (8.14)$$

Introducing the transformation

$$\xi = \left(\frac{\beta_K^{3/2}s^3}{U^{*1/2}g'^{1/2}}\right)^{1/2}\bar{X}, \quad (8.15)$$

equation (8.13) becomes

$$\left(\frac{\partial\bar{\psi}}{\partial\xi}\right)^2 = 2[\varepsilon + \delta\bar{\psi} + 2\bar{y}\bar{\psi} - \bar{y}^2] \\ \{[\varepsilon + \delta\bar{\psi} + \bar{\psi}^2]^{1/2} - [\varepsilon + \delta\bar{\psi} + 2\bar{y}\bar{\psi} - \bar{y}^2]^{1/2}\} \quad (8.16)$$

In this form all the parameters of the problem appear in the two dimensionless groupings ε and δ only. The meaning of δ is evident. The significance of ε is readily seen from equation (8.6). In terms of the dimensionless variables we have

$$D^2 = D^{*2} + \frac{2\,(f_K + \beta_K s\bar{y})}{g'}\,U^* s\bar{\psi} - \frac{U^* \beta_K s^2 \bar{y}^2}{g'} \tag{8.17}$$

Thus $U^*\,\beta_K s^2/g'$ is a measure of the change of the square of the depth along a streamline and $\varepsilon^{1/2}$ is the ratio of the characteristic depth D^* to the characteristic change of depth.

In Section 7 the equations were applied to the homogeneous ocean by neglecting the variation in depth. Hence, it is to be expected that equation (8.16) will yield the solution for the homogeneous ocean when $\varepsilon \gg 1$; i.e. as far as the boundary region is concerned, a two-layer model, the depth of whose upper layer is very great compared to the change of that depth, behaves like a homogeneous model. The approximate analytical solution for $\varepsilon \gg 1$ is obtained in the following manner. The first bracket on the right-hand side of equation (8.16) is approximated by ε. The expressions inside each of the other brackets are divided by ε and the brackets expanded in binomial series retaining terms of order ε^{-1}. This yields

$$\left(\frac{\partial\bar{\psi}}{\partial\xi}\right)^2 \approx \varepsilon^{1/2}(\bar{y} - \bar{\psi})^2 \tag{8.18}$$

Imposing the boundary condition $\bar{\psi}\,(o, \bar{y}) = o$, the solution of equation (8.18) is

$$\bar{\psi} = \bar{y}(1 - e^{-1/4\,\xi}) \tag{8.19}$$

which becomes identical with the solution obtained previously (equation 7.17) upon transformation to the appropriate variables.

If we deal with a homogeneous ocean, ε will in general be large because $g' = g$ and this quantity is of the order of 500 times as great as g' in a usual two-layer model. If we deal wthi a two-layer model ε may be made large by making D^* large. The preceding analysis

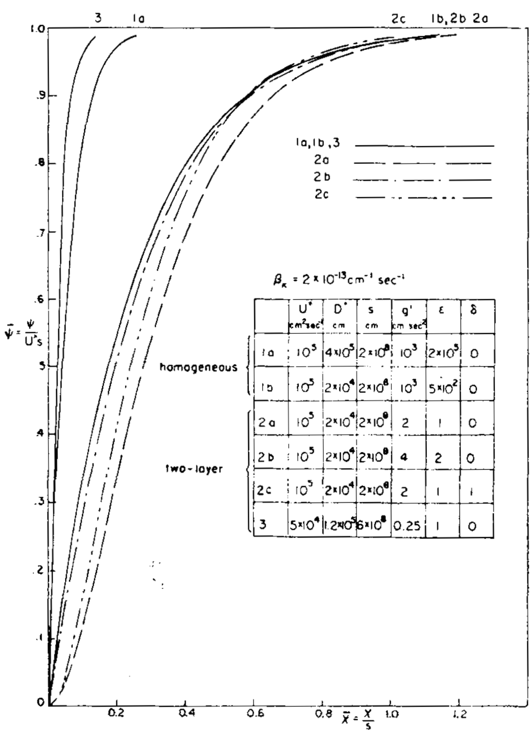

Fig. 2. Dimensionless transport function $\bar{\psi}$ vs. dimensionless distance from western coast at northern boundary of region I_b for various values of the pertinent paramaters.

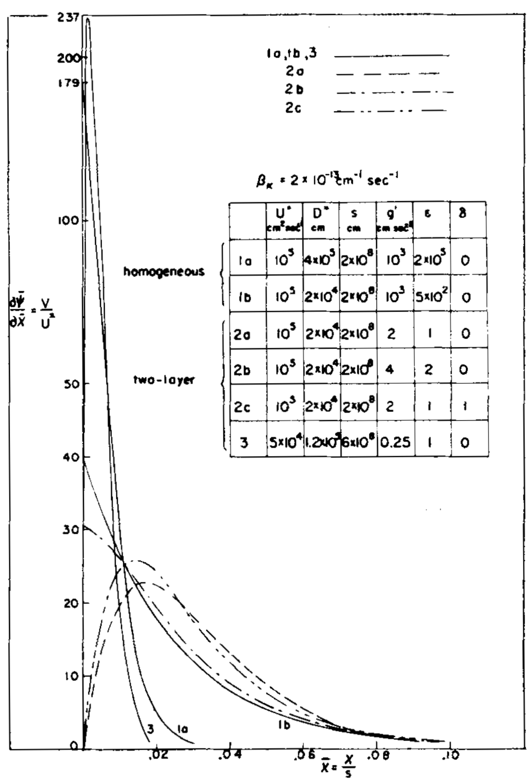

Fig. 3. Dimensionless northward transport vs. dimensionless distance from western coast at northern boundary of region I_b for various values of the pertinent parameters.

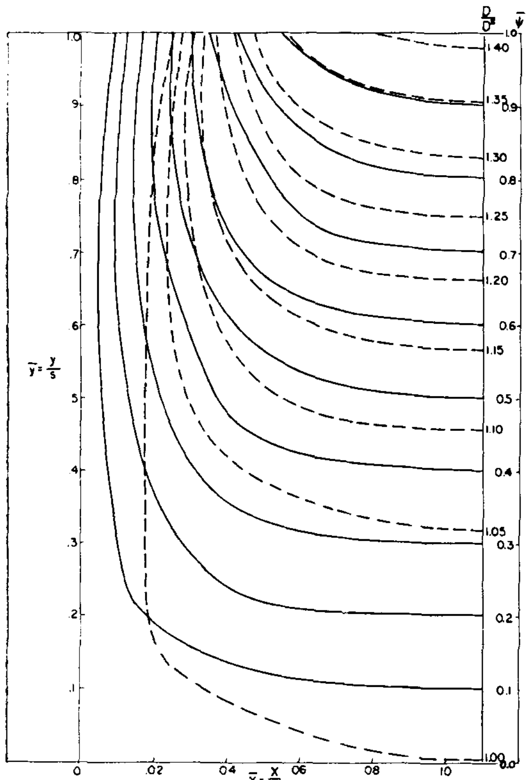

Fig. 4. Constant depth lines (dashed) and "transport" lines (solid) in region I_b for case 2 a of Figs. 2 and 3 corresponding approximately of the formation of the Gulf Stream.

shows that in terms of the variables of equation (8.16) the boundary solutions $\bar{\psi}$ for a homogeneous ocean and a two-layer model are identical provided ε is large and has the same value in the two cases.

Equation (8.16) has been solved numerically for various values of ε and δ and pertinent results are plotted in Figures 2 to 4.

Figure 2 shows the variation of the dimensionless transport function $\bar{\psi}$ with dimensionless distance $\bar{X}$ from the coast at the northern boundary of region I_b. The magnitudes of β_K, U^*, s, g', D^*, ε (see legends of Figs. 2, 3) for Curve 2 a correspond to a baroclinic model of the Gulf Stream except for the fact that the southern boundary of the region I is at the equator ($\delta = 0$). The width of the Stream is, say, 150 km, a very reasonable value. Curve 2 c represents the same situation with $\delta = 1$, i.e. with the southern boundary at approximately 15° latitude. This northward shift of the southern boundary tends to produce a somewhat narrower stream, but the effect is quite small. Curve 1 b corresponds to a homogeneous ocean with the values of U^*, D^*, s and δ equal to those of the upper layer of the two-layer model of Curve 2 a. These curves thus afford a revealing comparison of the behavior of the two models. The widths of the Streams are practically the same. Thus, the question of whether the bottom surface is solid, or an interface between two layers of slightly different density in a baroclinic model, does not have much bearing on the formation of the Stream and its width. Another interesting comparison may be made between Curves 2 a and 1 a, the latter (see Section 7) corresponding to a homogeneous ocean whose depth is that of the total two-layer model with the same values of U^*, s, and δ and which therefore represents the Stream that would exist if the motion were barotropic.

Curve 2 b represents the same situation as 2 a *except that the density difference is* double. Comparison of the two curves demonstrates the relative insensitivity of the stream formation to this factor.

Curve 3 applies to an ocean with the same values of ε and δ as 2 a but different U^*, s, g', D^*. The drastic decrease in the width of the current may be seen from equation (8.15) to be due primarily to the increase in s.

In Figure 3 the dimensionless transport $\partial\bar{\psi}/\partial\bar{X}$ is plotted against $\bar{X}$ for the systems discussed above. The transport shows a general tendency to decrease monotonically with $\bar{X}$ due to the decrease in velocity. This tendency, however, is counterbalanced by the monotonic increase of the depth, so that in some cases the transport first increases to a maximum and then decreases. This is the case in 2 a, 2 c and 3 where the transport starts from zero due to the fact that the depth vanishes at $\bar{y} = s$, $\bar{X} = 0$ in these cases; (see the following discussion of this point). The narrow streams of cases 1 a and 3 give rise to very large transports extending over a very narrow region. The transport adjacent to the coast in the homogeneous model 1 b is much greater than that in the corresponding baroclinic model 2 a due to the latter's very small depth near the coast. The transport in 1 b drops off much more rapidly with $\bar{X}$, however, so that the widths of the streams are about equal (Fig. 2).

Figure 4 represents a typical pattern of transport and constant depth lines (case 2 a). The transport lines near the coast tend to bend away from the latter after first approaching it due to the decrease in depth near the coast. This decrease tends to cut down the transport near the coast and hence to force the stream seawards.

Our results indicate that the density stratification influences certain aspects of the stream formation, but that it is a modifying factor rather than a fundamental one insofar as our problem is concerned. One important point must be discussed in this connection, however.

Equation (8.17) shows that, for fixed y, the depth is smallest at the boundary and that it becomes zero when

$$\bar{y} = \varepsilon^{1/2} \quad \text{or} \quad y^2 = \frac{D^{*2} g'}{U^* \beta_{KS}} \qquad (8.20)$$

Thus, if $\varepsilon < 1$, the solution cannot be valid for $\bar{y} > \varepsilon^{1/2}$ and the value $\bar{y} = \varepsilon^{1/2}$ might be interpreted as the latitude north of which a new regime of flow must take over. Since the velocity remains non-zero at that point we must actually except that the solution will break down at some distance south of $\bar{y} = \varepsilon^{1/2}$.

The preceding remarks apply to the behavior of the boundary solution when an arbitrarily fixed transport U^* with arbitrary depth D^* at $y = 0$ flows into the boundary region. If, however, we require that the boundary solution be matched to an interior solution the validity of which is to extend up to the eastern shore, then D^* and U^* are not independent. From equation (8.2)

$$C(L) = D^{*2} \approx D_a^2 + \frac{2\, Wa \sin K}{\varrho g'} \qquad (8.21)$$

whence, using the definitions of ε and U^x,

$$\varepsilon = \frac{D^{*2}}{D^{*2} - D_a^2} \qquad (8.22)$$

Hence, in this case $\varepsilon \geqslant 1$ and takes on its smallest value, one, only in the extreme case when $D_a = 0$. Since the meridional transport is independent of x in the interior, this extreme case would imply infinite meridional velocity at $x = a$.

It is worth noting that when the model under investigation does not represent an enclosed ocean, but rather a system in which northward flow is allowed to enter the stream across the boundary $y = 0$ so that the value of ψ on the western shore is negative, then, from equation (8.17), the depth will become zero for a value of $\bar{y}$ which is smaller than $\varepsilon^{1/2}$, so that the solution may break down for $\bar{y} < 1$ even if it is matched to an interior solution which is valid right up to $x = a$. It appears that this breakdown of the solution when ε is sufficiently small constitutes the major difference in the dynamics of the streams in a two-layer and a homogeneous model.

The theory predicts that the meridional velocity is a maximum at the western shore. This appears to violate completely the condition that a viscous fluid should adhere to a solid boundary, a condition which was specifically excluded by our approximate analysis. This violation becomes less serious than appears at first sight when one considers that the boundary may be interpreted as a water boundary with the region between the stream and the coast acting as sub-layer similar to the laminar sub-layer encountered in turbulent boundary layer flows.

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