Global Radiation Resulting from Multiple Scattering in a Rayleigh Atmosphere*

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Abstract

Using Chandrasekhar's solution for the radiative transfer problem in a plane-parallel model of the earth's atmosphere, under the assumption of perfect, conservative scattering according to the Rayleigh law, expressions are derived for the relative global and sky radiation received on a horizontal surface, as a function of normal optical thickness and inclination of incoming parallel radiation. The corrections representing the effect of ground reflection are also expressed in terms of certain functions already defined by Chandrasekhar. These expressions, which include the effect of *all orders of scattering*, are then used to compute the absolute global and sky radiation as a function of wavelength and solar zenith distance, based on the solar extraterrestrial energy curve as given recently by Nicolet. The results show that, under these assumptions, the global radiation received at the surface of the earth should remain essentially constant in spectral distribution for a wide range of solar altitudes, and that the pure sky radiation should have two maxima, centered at 3,300 Å and 4,100 Å respectively, for solar zenith distances ranging between 0° and 45°.

The theoretical work is compared with Bernhardt's results obtained by means of certain simplifying assumptions about the successive orders of scattering.

The absolute global and sky radiation is integrated over the range 0.29 μ to 4.00 μ, and it is shown that in general the computation is in good agreement with existing measurements under clear sky conditions.

1. Introduction

A knowledge of the amount and of the spectral distribution of the radiation received by the earth's surface is essential in all problems dealing with the radiational effects in the atmosphere. Several attempts have been made in the past to evaluate this quantity from theoretical considerations. The main difficulty encountered in all of these arises from the complicated nature of the radiative transfer in the atmosphere, which, in the most general case, is rather inaccessible to mathematical solutions in explicit form. Thus the efforts have been mainly limited to the evaluation of this radiation in certain simplified models of the earth's atmosphere, assuming Rayleigh scattering only. But even in this case the diffi-

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culties connected with an exact evaluation of the effects of scattering of orders higher than the first are such that only the secondary scattering could be included so far (cf. L. V. KING, 1913) and even that in the form of approximative solutions only.

Only very recently however, an ingenious method developed by S. CHANDRASEKHAR, (1950) especially suited to problems of radiative transfer in stellar and planetary atmospheres, has made it possible not only to incorporate, in an exact solution, all orders of scattering, but also to introduce the effects of ground reflection. In the present study, Chandrasekhar's exact solution is used in order to derive an expression for the total solar energy received by a horizontal unit area of the earth's surface, in such a way that the relative importance of the sky radiation and the radiation due to earth reflection can be appreciated accurately, as a function of wave length and sun elevation. Numerical computations are then carried out using this expression and the scattering functions involved, which were available to the authors (SEKERA, BLANCH, 1952; SEKERA, ASHBURN, 1953). Finally, these results are applied to a recent determination of the sun's extraterrestrial energy curve in order to obtain—perhaps for the first time—the exact theoretical spectral distribution of solar energy reaching the earth's surface after passage through a perfectly scattering atmosphere. As will be seen below, the theory, in so far as it can be compared with observation, gives remarkably consistent and accurate results, notwithstanding the highly idealized model used.

2. Formulation of the problem and its theoretical solution

Because of its small depth as compared to the earth's radius, for the purposes of our problem the atmosphere may be represented by a plane parallel one of infinite lateral extent but of finite depth. The density will be assumed to depend only on height above the ground, varying in accordance with mean conditions in the actual atmosphere. Furthermore, because of the large distance of the sun from the earth, the solar radiation will be represented by *parallel* radiation, of a known spectral distribution, incident on the topmost layer of the atmosphere. The problem then consists in finding the *inward flux* of radiant energy through a unit area of the earth's horizontal surface, as a function of the zenith distance of the sun, and of the wavelength and intensity of the incident radiation.

The problem can be simplified by separating this net flux into two distinct types of radiation: (a) a flux of parallel, direct solar radiation, modified only by the transfer through the atmosphere, called *sun radiation;* (b) a flux of non-parallel, diffuse radiation originating from the various scattering centers of the atmosphere, such as gas molecules, dust particles, etc., coming from the whole sky and called *sky radiation.* At present this sky radiation can be computed only under the assumptions (i) that the scatterers are of negligibly small dimensions when compared to the wavelength of the radiation, (ii) that all the radiation received by the scatterers is reradiated in all directions without modification of the original wavelength (i.e., case of conservative perfect scattering). These are the assumptions under which the well known Rayleigh theory of scattering (STRUTT, 1871; RAYLEIGH, 1899), is applicable, which provides the laws governing the intensity of the scattered light as a function of direction and wavelength as well as its state of polarization.

The sun radiation is proportional to the so-called reduced flux. The incident solar energy is simply assumed to be attenuated or reduced by the amount scattered *in all directions* by the scatterers, without, however, losing its character of a parallel radiation. If $\pi F_{0\lambda}$ be the net flux of monochromatic radiation of wavelength λ, incident at the top of the atmosphere under the zenith angle Θ_0, then the reduced flux πF_λ through a unit area normal to the incoming radiation, at a height z_1 above the earth's surface, can easily be shown to be

$$\pi F_\lambda = \pi F_{0\lambda} e^{-\tau/\mu_0} \tag{1}$$

where $\mu_0 = \cos \Theta_0$; the non-dimensional parameter τ, called the *normal optical thickness*, a function of the wave length λ, stands for the integral

$$\tau = \int_{z_1}^{\infty} \beta_\lambda dz \tag{2}$$

where β_λ is the monochromatic attenuation coefficient, a function of the density and of the refractive index of the air, and z is the distance measured vertically above the earth. For a given distribution of air density with height, the normal optical thickness (2) corresponding to Rayleigh scattering can be determined (DEIRMENDJIAN, 1952). Then, at any fixed level z cm above sea level, the optical thickness becomes a function of the wavelength only, and can therefore serve as the dimensionless parameter defining the wavelength of the incident radiation.

The net flux of parallel radiation through a horizontal unit area of the earth's surface, i.e. the *sun radiation* at the wavelength λ, will be given by the expression

$$S_\lambda = \pi F_\lambda \mu_0 = \pi F_{0\lambda} \mu_0 e^{-\tau/\mu_0} \tag{3}$$

which reduces to (1) when the sun is exactly at the zenith ($\Theta_0 = 0$).

The *sky radiation*, as defined above, represents the flux of the radiation scattered by all the scattering centers in the atmosphere. Not only is the direct solar radiation itself scattered, but also part of all the radiation reaching the earth's surface which is reflected back into the atmosphere. As will be seen below, under certain assumptions this reflection will result in an increase in sky radiation which can be computed separately.

Considering first the standard problem, i.e., the one without reflecting boundaries, the monochromatic radiation received from an elementary cone of solid angle $d\omega$ and from a direction making the angle Θ with the vertical will contribute to the net flux through a horizontal unit area by the amount $I \cos \Theta\, d\omega$, where in general, $I = I\,(\Theta_0, \varphi_0, \Theta, \varphi)$ is the *specific intensity of the diffuse sky radiation*. The sky radiation is then given by the integral

$$H_\lambda = \int I \cos \Theta d\omega \tag{4}$$

where the integration is to be performed over all the solid angles of the *outer hemisphere only*, i.e., it is assumed that the surface element is receiving radiation from one side only. Since $d\omega = \sin \Theta d\Theta d\varphi$ (φ denotes the azimuth of the particular direction), the expression (4) can also be written as

$$H_\lambda = \int_0^1 \int_0^{2\pi} I \mu\, d\mu\, d\varphi, \quad (\mu = \cos \Theta) \tag{5}$$

The diffuse sky radiation is in general polarized and thus the specific intensity I can be expressed as the sum of the intensities of two components polarized in two directions perpendicular to each other, for example, normal (I_r) and parallel (I_l) to the vertical plane through the direction given by the zenith angle Θ and the azimuth φ. From the general equations of radiative transfer (with all orders of scattering included) according to the method described by CHANDRASEKHAR (1950, 1951), the expression for the intensities I_l and I_r may be written in the form

$$I_i = I_i^0 + A_i(\mu, \mu_0) \cos(\varphi - \varphi_0) + + B_i(\mu, \mu_0) \cos 2 (\varphi - \varphi_0), \quad (i = l, r) \tag{6}$$

where μ_0 and φ_0 specify the direction of the incident radiation and the functions A_i and B_i are linear combinations of six special functions of μ and μ_0 only (CHANDRASEKHAR, 1950; SEKERA, BLANCH, 1952). If the expression (6) is substituted for the components of I in (5), i.e. placing $I = I_l + I_r$, it is evident that the azimuth dependent terms—i.e. the terms containing $\cos(\varphi - \varphi_0)$ and $\cos 2(\varphi - \varphi_0)$—will vanish upon integration with respect to φ. Thus only the *azimuth independent* functions I_l^0 and I_r^0 of (6) will contribute to the integration in (5), as might be expected from symmetry considerations. These functions I_i^0 depend on the flux of the incident radiation according to the equation (cf. CHANDRASEKHAR, 1950, pp 265, 276)

$$\begin{pmatrix} I_l^0 \\ I_r^0 \end{pmatrix} = \frac{3}{16\,\mu} \boldsymbol{T}^0(\mu, \mu_0) \begin{pmatrix} F_l \\ F_r \end{pmatrix} \tag{7}$$

where $\boldsymbol{T}^0$ represents the *transmission matrix* with elements T_{ik}^0 $(i, k = l, r)$. Hence substitution of (7) into (5), with the assumption of a *neutral* incident radiation, i.e. $F_l = F_r = F_0/2$, yields the explicit integral for the sky radiation

$$H_\lambda = \frac{3\,\mu F_0}{16} \int_0^1 (T_{ll}^0 + T_{lr}^0 + T_{rl}^0 + T_{rr}^0)\, d\mu \tag{8}$$

or with the use of the functions γ_l and γ_r, conveniently introduced by CHANDRASEKHAR (1950, p. 276),

$$H_\lambda = \pi F_0 \mu_0 \left[\frac{1}{2} \gamma_l(\mu_0) + \frac{1}{2} \gamma_r(\mu_0) - e^{-\tau/\mu_0}\right] \tag{9}$$

A comparison with (3) shows immediately that the expression

$$\frac{1}{2}\pi F_0\mu_0\,[\gamma_l(\mu_0) + \gamma_r(\mu_0)] = H_\lambda + S_\lambda \quad (10)$$

represents the monochromatic *global* (sun + sky) radiation where the functions $\gamma_l(\mu_0)/2$ and $\gamma_r(\mu_0)/2$ have a clear physical meaning: *They represent the monochromatic global radiation resulting from the components polarized parallel and normal respectively to the vertical plane, for unit incident flux* ($\pi F_0 = 1$) *of neutral radiation, when there is no reflection and* $\Theta_0 = 0$.

In the expressions (9) and (10) the effect of the ground reflection has not been taken into consideration. In general, such a reflection is equivalent to an additional illumination of the atmosphere from below, hence a corresponding increase of the diffuse sky radiation should be expected. In the special case of reflection according to Lambert's law*, this increase can be expressed in terms of the same γ_i functions mentioned above. The additional intensity $I_r^\star + I_l^\star$ of diffuse light due to the ground reflection is also azimuth independent and has the form (CHANDRASEKHAR, ELBERT, 1951)

$$I_l^\star + I_r^\star = \frac{A\mu_0F_0}{4(1-A\bar{s})}[\gamma_l(\mu_0)+\gamma_r(\mu_0)]\,[2-\gamma_l(\mu)-\gamma_r(\mu)] \quad (11)$$

where A is the albedo and $\bar{s}$ is a quantity depending upon τ only, but independent of μ_0, connected with the γ_i functions by the relation (cf. CHANDRASEKHAR, 1950, p. 280)

$$\bar{s} = 1 - \int_0^1 [\gamma_l(\mu)+\gamma_r(\mu)]\,\mu d\mu \quad (12)$$

The sky radiation resulting from ground reflection only, can now be easily computed by using the expression (11) for the intensity I in (5). The integration, carried out by using (12) for $\bar{s}$, yields the result

$$\int_0^1\int_0^{2\pi}(I_l^\star + I_r^\star)\,\mu d\mu d\varphi = \frac{\pi F_0\mu_0 A\bar{s}}{2(1-A\bar{s})}\{\gamma_l(\mu_0) + \gamma_r(\mu_0)\} \quad (13)$$

and if this is added to (9) one obtains a simple expression for the sky radiation which includes the effect of the ground reflection, namely

$$H_\lambda = \pi F_0\mu_0\left\{\frac{\gamma_l(\mu_0) + \gamma_r(\mu_0)}{2(1-A\bar{s})} - e^{-\tau/\mu_0}\right\} \quad (14)$$

Finally, when (14) is in turn added to the expression (3) for the sun radiation, a still simpler expression results, representing the monochromatic global radiation G_λ when the earth is reflecting with an albedo A, i.e.,

$$G_\lambda = H_\lambda + S_\lambda = \frac{1}{2}\pi F_0\mu_0\,\frac{\gamma_l(\mu_0) + \gamma_r(\mu_0)}{1-A\bar{s}} \quad (15)$$

Thus, the global radiation can be easily computed provided the functions $\gamma_l(\mu)$ and $\gamma_r(\mu)$ are known. The evaluation of these functions is not so simple, however, requiring a solution of two systems consisting each of two non-linear integral equations for two pairs of functions (X_l, Y_l, X_r, Y_r in Chandrasekhar's notation). The solution of these equations can be obtained by successive iterations, and has been performed (SEKERA, BLANCH, 1952) for three normal optical thicknesses, namely $\tau = 1.00$, 0.25, and 0.15 corresponding at sea level to the wavelengths 0.312 μ, 0.436½ μ, and 0.495 μ respectively. During the computation, it was found that the corrected second approximation, corresponding physically to the inclusion of secondary scattering only, gives values of sufficient accuracy for $\tau \leqq 0.10$, so that the γ_i functions in this range were computed without iteration by direct analytical computation (FRASER, 1952, 1953; SEKERA, ASHBURN, 1953). For even smaller values of τ (i.e. $\tau \leqq 0.01$) when the second approximation leads to inaccurate numerical computations, the first approximation corresponding to the primary scattering only provides a sufficient accuracy. In such a case the expressions for the intensities I_l and I_r can be easily derived directly from the equation of radiative transfer (cf. CHANRASEKHAR, 1950 p. 44), which for the primary scattering reduces to the form

$$\mu\frac{d\boldsymbol{I}}{d\tau} = \boldsymbol{I}(\tau,\mu,\varphi) - \frac{1}{4}e^{-\tau/\mu_0}\boldsymbol{P}(\mu,\varphi;\mu_0,\varphi_0)\boldsymbol{F} \quad (16)$$

where $\boldsymbol{I}$ and $\boldsymbol{F}$ denote one column matrices of Stokes parameters for the intensity of diffuse

* i.e. the reflected radiation is assumed to be unpolarized and independent of direction, its total outward flux being always a constant fraction A, ($0 \leqq A < 1$) of the incoming flux, called the *albedo* of the reflecting surface.

sky radiation and for the flux of extraterrestrial incident radiation, respectively, and $\boldsymbol{P}$ denotes the phase matrix for Rayleigh scattering [cf. CHANDRASEKHAR, 1950, p. 42, equ. (220)]. For a neutral incident radiation ($F_l = F_r = \frac{1}{2} F_0$), if in (16) only the azimuth independent terms are considered, the equations for I_l^0 and I_r^0 asume the form

$$\mu \frac{dI_i^0}{d\tau} = I_i^0 - J_i(\tau, \mu), \quad (i = l, r) \tag{17}$$

where

$$J_l(\tau, \mu) = \frac{3}{32} F_0 e^{-\tau/\mu_0} [2(1-\mu^2)(1-\mu_0^2) + \\ + \mu^2(1+\mu_0^2)]$$

$$J_r(\tau, \mu) = \frac{3}{32} F_0 e^{-\tau/\mu_0} [1+\mu_0^2]$$

The solutions of (17), satisfying the boundary condition

$$I_i^0(0, -\mu, \varphi) = 0, \quad (i = l, r)$$

can be easily computed, and when added, the following expression results:

$$I_l^0 + I_r^0 = \frac{3}{32} \mu_0 F_0 [3 - \mu_0^2 + \\ + (3\mu_0^2 - 1)\mu^2] \frac{e^{-\tau/\mu_0} - e^{-\tau/\mu}}{\mu_0 - \mu} \tag{18}$$

The sky radiation $H_\lambda^{(1)}$ corresponding to primary scattering only is then given by (5) with the expression (18) substituted for I. The integration yields the final result in the form

$$H_\lambda^{(1)} = \frac{3}{16} \pi F_0 e^{-\tau/\mu_0} \{(3-\mu_0^2) F_2(\tau, \mu_0) + \\ + (3\mu_0^2 - 1) F_4(\tau, \mu_0)\} \tag{19}$$

where the functions F_2, F_4 are of the form

$$F_i(\tau, \mu_0) = \mu_0 \int_0^1 \frac{\mu^{i-1}}{\mu_0 - \mu} \left\{ 1 - \exp\left[-\tau\left(\frac{1}{\mu} - \right.\right.\right. \\ \left.\left.\left. - \frac{1}{\mu_0}\right)\right]\right\} d\mu \tag{20}$$

$$(i = 1, 2, 3, 4)$$

and related closely to the exponential integral. (Cf. CHANDRASEKHAR, 1950, App. I, p. 375.)

There is no difficulty involved in the computation of these functions since they are given by very simple recurrence formulae. For very small τ the functions F_i can be easily computed from the series expansions in powers of τ. The first term in these expansions gives the approximation

$$F_i = \frac{\tau}{i-1} \qquad (i = 2, 3, \ldots)$$

so that (19) reduces to the very simple form

$$H_\lambda^{(1)} = \frac{1}{2} \pi F_{0\lambda} \tau \tag{21}$$

valid for $0 < \tau << 1$.

Equations (19) and (21) give the sky radiation for the case of zero albedo. To get the correction term in the case of reflection, one must add to this the corresponding sun radiation (3) and multiply the sum by the factor $A\bar{s}/(1 - A\bar{s})$ appearing in (13). If greater accuracy is desired, one can examine the behavior of the ratio $H_\lambda^{(1)}/H_\lambda$ as $\tau \to 0$, which should approach unity asymptotically, and then correct the $H_\lambda^{(1)}$ values accordingly.

The expressions (3), (14), (15), and (19) for the sun, sky, and global radiation contain the monochromatic solar extraterrestrial flux $\pi F_{0\lambda}$ as a constant factor. If this flux be taken as unity, then the above expressions represent what is sometimes known as the *relative* fluxes. On the other hand if the actual extraterrestrial flux is substituted for $\pi F_{0\lambda}$ then the expressions (3), (14), (15), (19) will represent the *absolute* fluxes of sun, sky, and global radiation.

Because of the instrumental difficulties in the measurement of the sky and sun radiation within narrow spectral ranges, almost all the measurements of these quantities, so far available, have been made by means of a bolometric instrument registering the total radiation in all wavelengths. The measured values can thus be compared with quantities derived from the absolute fluxes of the sky, sun or global radiation by integration over all wavelengths. These quantities will be denoted by S, H or G respectively, with

$$S = \int_0^\infty S_\lambda d\lambda, \quad H = \int_0^\infty H_\lambda d\lambda, \quad G = \int_0^\infty G_\lambda d\lambda \tag{22}$$

and called the integrated sun, sky, and global radiation.

3. Numerical results

Relative Fluxes. The relative fluxes S_λ, H_λ, and G_λ were computed first and the result of the computation is given in Tables I, II, and III. The tables cover four values of the variable μ_0, i.e. 1.00, 0.60, 0.10, and 0.02 corresponding respectively to a solar zenith distance of 0°.0, 53°.1, 84°.3 and 88°.8; and twelve values of τ, the normal optical thickness. The particular value of $\mu_0=0.02$ ($\Theta_0=88°.8$) was chosen for the following reason: In an infinite plane-parallel atmosphere, sunrise and sunset conditions would correspond to a zenith distance of 90°, in which case the theoretical S_λ and H_λ would be entirely unrealistic, on account of the complete attenuation of the incident radiation (infinite optical path). However, from the dioptrical tables of LINK and SEKERA (1940), the air mass and thus the attenuation factor along the light path in the actual spherical atmosphere, when the sun is at the local horizon, can be easily evaluated. These tables also take into account the bending of the parallel radiation caused by refraction. The attenuation arrived at in this way should then be considered as the maximum finite attenuation of the incident radiation in the plane-parallel model. The corresponding zenith distance turns out to be very close to 88°.8, which therefore is the value to be used in a plane-parallel atmosphere, whenever conditions equivalent to sunrise and sunset in the spherical atmosphere are to be approximated.

The parameter τ is used in preference to the wavelength for obvious reasons of general applicability. The following list shows the wavelengths corresponding at *sea level* to the values of τ used in this paper, as obtained from a recent computation (DEIRMENDJIAN, 1952):

τ	1.00	0.25	0.15
λ at sea level (microns)	0.312	0.436½	0.495
τ	0.06	0.02	8.5×10^{-3}
λ at sea level (microns)	0.618	0.809	1.000
τ	16.5×10^{-4}	5.2×10^{-4}	2.15×10^{-4}
λ at sea level (microns)	1.500	2.000	2.500
τ	1.03×10^{-4}	5.6×10^{-5}	3.25×10^{-5}
λ at sea level (microns)	3.000	3.500	4.000

The values for S_λ can be computed easily from a table of the exponential function. The computations for G_λ and H_λ were carried out as outlined in the previous section, i.e. in the range $0.15 \leqq \tau \leqq 1.00$ the γ functions for multiple scattering were used as given by the Tables (SEKERA, BLANCH, 1952); for $\tau=0.06$ and 0.02 the analytically computed γ_i functions (FRASER, 1952, 1953; SEKERA, ASHBURN, 1953) were used, implying the inclusion of secondary scattering only; and for range $3.25 \times 10^{-5} \leqq \tau \leqq 8.5 \times 10^{-3}$ the quantity $H_\lambda^{(1)}$ was computed directly from (19) and then corrected by using the ratios $\frac{H_\lambda^{(1)}}{H_\lambda}$ shown below: The figures in brackets indicate the extrapolation for lower τ's, justified both by physical considerations and by the fact that the quantity $\log\left(1-\frac{H_\lambda^{(1)}}{H_\lambda}\right)$ becomes a linear function of $\log \tau$ when $\tau \to 0$. These values of $\frac{H_\lambda^{(1)}}{H_\lambda}$ are of particular interest since, for the first time, they exhibit exactly the contribution of the primary scattering to the sky radiation from all

$H_\lambda^{(1)}/H_\lambda$ (albedo = 0)

μ_0 \ τ	1.00	0.25	0.15	0.06	0.02	8.5×10^{-3}	16.5×10^{-4}	5.2×10^{-4}
1.00	.403	.734	.812	.904	.960	[.98]	[.99½]	[1.00]
0.60	.352	.699	.785	.887	.95	[.98]	[.99]	[1.00]
0.10	.271	.667	.765	.877	.95	[.97]	[.99]	[1.00]
0.025		.66	.75	.88	.95	[.97]	[.99]	[1.00]

Table I. The *relative sun radiation* S_λ as a function of normal optical thickness τ (see text for corresponding wavelengths) and cosine of the solar zenith distance

μ_0 \ τ	1.00	0.25	0.15	0.06	0.02	8.5×10^{-3}	16.5×10^{-4}	5.2×10^{-4}	2.15×10^{-4}	1.03×10^{-4}	5.6×10^{-5}	3.25×10^{-5}
1.00	.3679	.7788	.8607	.9418	.9802	.9915	$.9983^5$	.9995	.9998	.9999	.9999	1.0000
0.60	.1133	.3955	.4673	.5429	.5803	.5915	.5984	.5995	.5998	.5999	.5999	.6000
0.10	.0000	.0082	.0223	.0549	.0819	$.0918^5$	.0984	.0995	.0998	.0999	.0999	.1000
0.02	.0000	.0000	.0000	.0010	.0074	.0131	.0184	.0195	.0198	.0199	.0199	.0200

orders of scattering. It is seen that for infrared radiation of $\lambda \geqq 2.00$ there would be less than 0.5 % error if the secondary and higher order scattering is neglected.

As can be seen from Tables I and II, at any fixed sun elevation, while the relative sun radiation S_λ increases asymptotically to μ_0 with decreasing τ (increasing wavelength), the relative sky radiation *decreases* in the same direction, in such a way that for very small values of τ, H_λ is proportional to $\frac{1}{2}\tau$ as per (21), becoming independent of solar elevation. The sum of these two quantities, i.e. the relative global radiation G_λ is shown in Table III which, like Table II, is divided into three parts corresponding to no reflection and Lambert reflection with albedo 0.25 and 0.80 respectively. The relative global radiation generally increases with decreasing τ, but there is not as big a difference between short and long wavelengths as in the case of the sun radiation alone. For the case of an albedo of 0.80 and $\mu_0 = 1.00$ (sun at the zenith) the variation with τ is almost reversed, so that the maximum in G_λ occurs near $\tau = 0.25$, at which point it exceeds the corresponding extra-terrestrial flux by almost 4 %, and then *decreases* asymptotically to μ_0. However, when comparing this interesting result with conditions in the real atmosphere, it should be borne in mind that the model used involves a number of unrealistic assumptions: a plane parallel atmosphere of an infinite lateral extent with each point of its infinite lower boundary reflecting according to Lambert's law with the high albedo of 0.80.

These features of the relative sun and sky radiation given by the theory are illustrated

Table II. The *relative sky radiation* H_λ for the same values of the parameters as for Table I, showing also the values corrected for Lambert reflection with an albedo of 0.25 and 0.80

μ_0 \ τ	1.00	0.25	0.15	0.06	0.02	8.5×10^{-3}	16.5×10^{-4}	5.2×10^{-4}	2.15×10^{-4}	1.03×10^{-4}	5.6×10^{-5}	3.25×10^{-5}
						Albedo = 0.00						
1.00	.2918	.1096	.0694	.0291	.0099	.0043	.0008	.0003	.0001	$.0000^5$	.0000	.0000
0.60	.2117	.1006	.0659	.0285	.0099	.0043	.0008	.0003	.0001	$.0000^5$	.0000	.0000
0.10	.0296	.0415	.0371	.0223	.0090	.0041	.0008	.0003	.0001	$.0000^5$	.0000	.0000
0.02	.0050	.0078	.0086	.0090	.0063	.0035	.0008	$.0002^5$	.0001	$.0000^5$	.0000	.0000
						Albedo = 0.25						
1.00	.3748	.1514	.0980	.0422	.0146	.0064	$.0012^5$	.0004	.0001	.0001	.0001	.0000
0.60	.2525	.1239	.0822	.0363	.0127	.0056	.0010	.0003	.0001	.0001	.0001	.0000
0.10	.0334	.0438	.0389	.0234	.0095	$.0042^5$	.0008	.0003	.0001	.0001	.0001	.0000
0.02	.0056	.0082	.0089	.0091	.0064	.0035	.0008	$.0002^5$	.0001	.0001	.0001	.0000
						Albedo = 0.80						
1.00	.6590	.2589	.1675	.0724	.0225	.0110	$.0021^5$	.0007	.0003	.0001	.0001	.0000
0.60	.3925	.1839	.1221	.0540	.0190	.0084	.0016	.0005	.0002	.0001	.0001	.0000
0.10	.0461	.0498	.0434	.0258	.0104	$.0047^5$	.0009	.0003	.0001	.0001	.0001	.0000
0.02	.0078	.0091	.0095	.0095	.0065	.0036	.0008	.0002	.0001	.0001	.0001	.0000

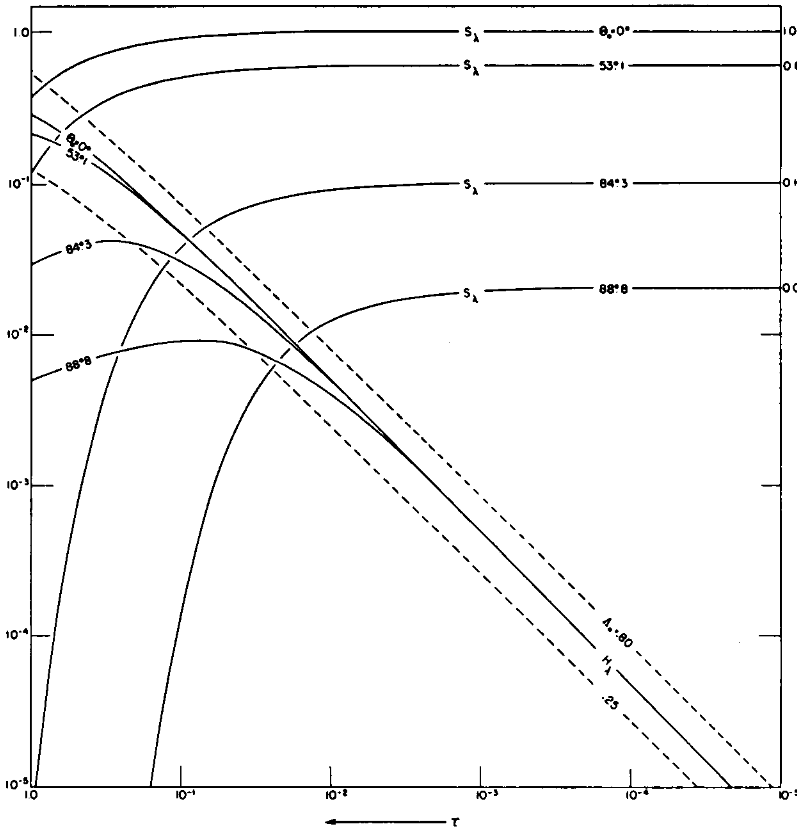

Fig. 1. Curves representing the variation of the log of relative sun (S_λ) and sky (H_λ) radiation for given solar zenith distance Θ_0, with the log of the normal optical thickness τ. The corresponding wavelengths would increase from left to right on the diagram. Dash lines represent the quantity $A\bar{s}/(1-A\bar{s})$ for albedo 0.25 and 0.80 (see text).

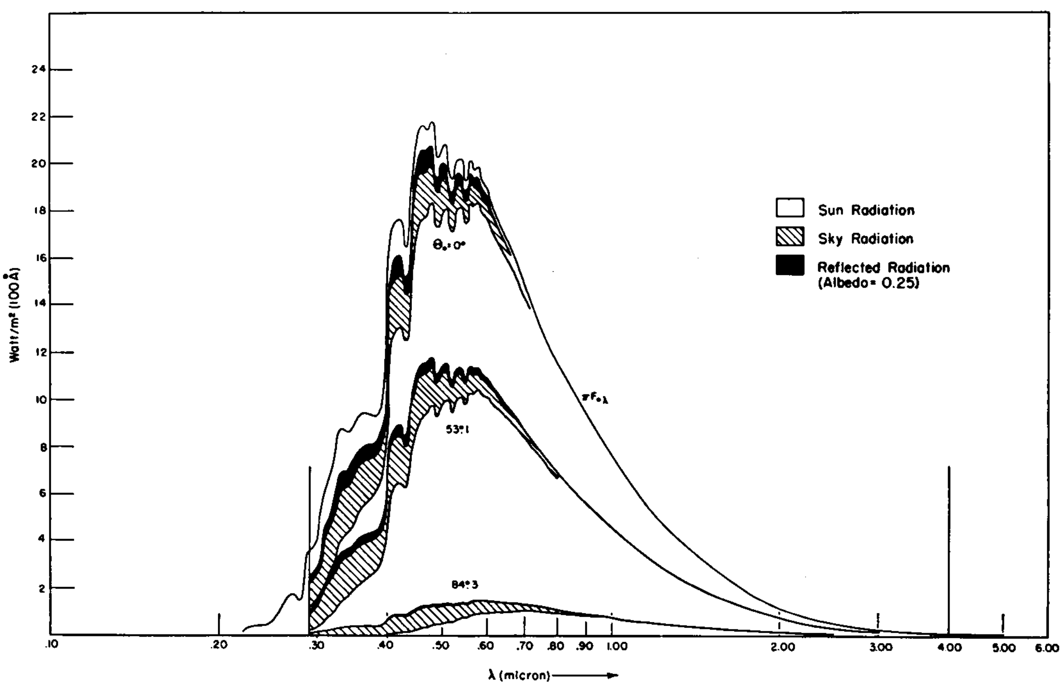

Fig. 2. Spectral distribution of solar extraterrestrial fluxes (topmost full line) as given by Nicolet, and resulting sun radiation on a horizontal surface at sea level, for different solar zenith distances. The hatched areas represent the additional radiation received from the whole sky due to multiple Rayleigh scattering, and the blackened areas, due to a Lambert reflection with albedo 0.25.

in the diagram of Fig. 1, which was drawn by fitting smooth curves to plots of log H_λ (albedo = 0) and log S_λ, versus log τ, taken from Tables I and II. The asymptotic behavior of H_λ as $\tau \rightarrow 0$ is clearly seen from the corresponding curves. Furthermore, *the maximum in relative sky radiation is seen to shift from shorter toward longer wavelengths as the zenith distance of the sun increases.*

Absolute Fluxes: It is well known that there is considerable uncertainity regarding the exact spectral distribution of the extraterrestrial flux of solar energy. A good analysis of up to date knowledge was presented recently by Marcel Nicolet (1948, 1951 a, 1951 b). There exist some more recent direct measurements of the solar spectrum, especially in the ultraviolet and near ultraviolet region, made with the use of spectrophotometers carried on rockets (JOHNSON ET AL., 1953), or by the Bouguer-Langley method applied to measurements from ground observatories (PETTIT, 1932; STAIR, 1952). These show certain minor differences from Nicolet's distribution, mainly in the near ultraviolet region. However, in the absence of more extensive and reliable rocket measurements, we have preferred Nicolet's figures which are consistent with present knowledge on the solar constant, and are sufficiently detailed for our purposes.

Nicolet gives values of the net flux of the radiation in watt per square meter per 100 Å, as received at the top of our atmosphere, covering the solar spectrum between 2,200 and 70,000 Å (see the uppermost curve of Fig. 2, marked $\pi F_{0\lambda}$). If we use the sea level wavelengths corresponding to the twelve values of τ used in the computation, the relative fluxes in Tables I, II, and III would cover the spectrum from 3,120 to 40,000 Å (this range was extended to 2,900 Å by a small extrapolation toward shorter wavelengths).

In order to obtain as detailed a spectrum of global and sky radiation as possible it was necessary to interpolate our numerical results for the small wavelength intervals used by Nicolet, i.e for intervals of 100 Å up to 10,000 Å. In the range $1\,\mu \leqq \lambda \leqq 4\,\mu$, we used values at intervals of 0.5μ as originally computed, since the solar energy curve is quite smooth in this region. The interpolation was carried out graphically by plotting H_λ on large scale logarithmic paper; by careful curve fitting it was possible to obtain interpolated values to three significant figures, sufficient for the desired accuracy. These interpolated values of H_λ added to the S_λ values computed from existing tables of e^x yielded the G_λ values representing the relative global radiation. A check was made by plotting G_λ curves and interpolating by the

Table III. The *relative global radiation* G_λ, tabulated on the same basis as Table II

μ_0 \ τ	1.00	0.25	0.15	0.06	0.02	8.5×10^{-3}	16.5×10^{-4}	5.2×10^{-4}	2.15×10^{-4}	1.03×10^{-4}	5.6×10^{-5}	3.25×10^{-5}
						Albedo = 0.00						
1.00	.6597	.8884	.9301	.9709	.9901	.9958	$.9991^5$	.9998	.9999	.9999	.9999	1.0000
0.60	.3250	.4961	.5332	.5714	.5902	.5958	.5992	.5998	.5999	.5999	.5999	.6000
0.10	.0296	.0497	.0594	.0772	.0909	$.0959^5$	.0992	.0998	.0999	.0999	.0999	.1000
0.02	.0050	.0078	.0086	.0100	.0137	.0166	.0192	$.0197^5$	.0199	.0199	.0199	.0200
						Albedo = 0.25						
1.00	.7427	.9302	.9587	.9840	.9948	.9979	.9996	.9999	.9999	1,0000	1,0000	1.0000
0.60	.3658	.5194	.5495	.5792	.5930	.5971	.5994	.5998	.5999	.6000	.6000	.6000
0.10	.0334	.0520	.0612	.0783	.0914	.0961	.0992	.0998	.0999	.1000	.1000	.1000
0.02	.0056	.0082	.0089	.0101	.0138	.0166	.0192	$.0197^5$	.0199	.0200	.0200	.0200
						Albedo = 0.80						
1.00	1.0269	1.0377	1.0282	1.0142	1.0054	1.0025	1.0005	1.0002	1.0001	1.0000	1.0000	1.0000
0.60	.5058	.5794	.5894	.5969	.5993	.5999	.6000	.6000	.6000	.6000	.6000	.6000
0.10	.0461	.0580	.0657	.0807	.0923	.0966	.0993	.0998	.0999	.1000	.1000	.1000
0.02	.0078	.0091	.0095	.0105	.0139	.0167	.0192	.0197	.0199	.0200	.0200	.0200

Table IV. The *absolute global radiation* $\pi(F_0G)_\lambda$ *in watt per* m^2 *per 100 Å* as a function of wavelength in microns and solar zenith distance Θ_0, based on Nicolet's values for $\pi F_{0\lambda}$

λ \ Θ_0	0°	53°.1	84°.3	λ \ Θ_0	0°	53°.1	84°.3
.29	2.14	1.00	0.10	.52	18.22	10.52	1.22
.30	2.59	1.25	.12	.53	19.07	11.03	1.31
.31	4.02	1.96	.18	.54	19.18	11.12	1.34
.32	4.90	2.44	.22	.55	18.39	10.71	1.31
.33	6.20	3.14	.28	.56	19.20	11.19	1.40
.34	6.35	3.26	.29	.57	18.99	11.09	1.41
.35	6.92	3.59	.32	.58	19.08	11.16	1.43
.36	7.33	3.85	.34	.59	18.53	10.86	1.41
.37	7.50	3.98	.36	.60	18.34	10.76	1.42
.38	7.56	4.06	.37	.65	16.42	9.70	1.35
.39	8.21	4.44	.41	.70	14.56	8.65	1.26
.40	13.06	7.13	.66	.75	12.87	7.65	1.15
.41	15.02	8.24	.78	.80	11.42	6.80	1.04
.42	15.33	8.47	.82	.85	10.38	6.19	.97
.43	14.47	8.05	.80	.90	9.36	5.59	.88
.44	17.01	9.50	.96	.95	8.45	5.05	.81
.45	18.82	10.56	1.09	1.00	7.65	4.58	.74
.46	19.59	11.04	1.16	1.50	2.98	1.79	.30
.47	19.65	11.15	1.19	2.00	1.19	.71	.12
.48	20.08	11.42	1.24	2.50	.47	.28	.05
.49	18.75	10.72	1.19	3.00	.26	.16	.03
.50	19.36	11.11	1.25	3.50	.16	.10	.02
.51	19.43	11.20	1.28	4.00	.10	.06	.01

same method. As for the albedo correction, the interpolation was obtained by means of a curve of the quantity $A\bar{s}/(1-A\bar{s})$ which closely parallels the H_λ curve for $\Theta_0=0°$, as can be seen in Fig. 1. The physical meaning of this quantity is evident from the expression (13); when multiplied by (10) it gives the additive correction to the sky radiation contributed by the ground reflection.

A sufficiently detailed spectrum of the absolute global $\pi(F_0G)_\lambda$ and sky radiation $\pi(F_0H)_\lambda$, as obtained by using Nicolet's values for $\pi F_{0\lambda}$, appears in Tables IV and V respectively for three solar positions*. It should be emphasized that these computations involve the assumption of *perfect scattering in a Rayleigh atmosphere*, hence the reader should bear this in mind when comparing with any actual observations. The irregularities appearing in the region of wavelengths shorter than 6,000 Å correspond to the Fraunhofer absorption regions indicated by Nicolet's data. A subtraction of corresponding values of Table V from those of Table IV yields values for the absolute sun radiation $\pi(F_0S)_\lambda$.

The corresponding values including the ground reflection with an albedo of 0.25 (considered rather high for mean planetary reflection) are not shown, but their variation with wavelength is similar to that of the sky radiation. The relative importance of each component of the global radiation can be readily seen from the diagram of Fig. 2, in which the wavelength scale is logarithmic. This shows the spectral distribution of sun, sky, and reflected radiation for the solar zenith distances of 0°, 53°.1 and 84°.3 respectively, as indicated. The lowermost curve in each case represents the absolute sun radiation, $\pi(F_0S)_\lambda$ received through unit horizontal area at sea level of the earth's surface. This radiation is poor in blue and ultraviolet light, and becomes even poorer in this respect the closer the sun is to the horizon. The next curve above, enclosing the hatched area, represents the contribution from the sky radiation, $\pi(F_0H)_\lambda$. This shows clearly how the light from the entire sky, according to the Rayleigh theory, restores, so to speak, most of the short wave radiation lost by attenuation

* The units of watt/m² were adopted for the net absolute flux following Nicolet. To convert this to Langleys or calories/cm², minute, multiply by the conversion factor 14.3335×10^{-4}.

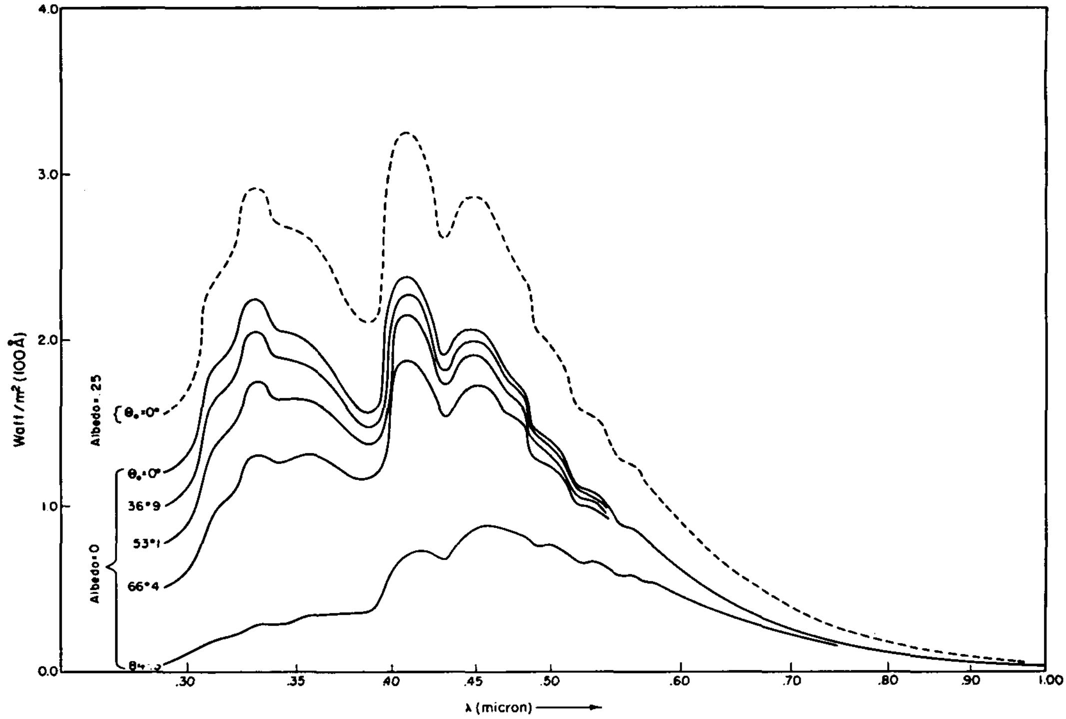

Fig. 3. Variation of the spectral distribution of sky radiation with solar zenith distance, based on the extraterrestrial flux curve of Fig. 2 and the Rayleigh-Chandrasekhar theory of multiple scattering. The full curves correspond to zero earth reflection and the dash curve shows the total sky radiation for a zenith sun in the case of reflection with albedo 0.25.

in the parallel radiation, so that the global radiation reaching the earth's surface is quite similar in spectral distribution (though not in magnitude) to the solar extraterrestrial curve. This similarity is conserved for a wide range of solar zenith distances, i.e. during the greater part of daylight hours. The additional sky radiation in the case of earth reflection with an albedo of 0.25, represented by the blackened area on top of the hatched area, is seen to be similar in distribution to the sky radiation.

The theoretical spectrum of the absolute sky radiation and its variation with solar position appears in the diagram of Fig. 3. This is drawn on the same basis as that of Fig. 2, except that the flux scale has been expanded. In the family of curves representing the case without earth reflection, the one corresponding to the sun at the zenith shows two pronounced maxima at about 3,300 and 4,100 Å respectively and a secondary one at about 4,500 Å, with minima in between which look like broad absorption regions. This feature of the sky radiation profile results from the character of the solar extraterrestrial flux curve (uppermost curve of Fig. 2) and the relative sky radiation values computed in the present work (Table II). It is seen from Fig. 3 that the maximum flux of sky radiation gradually shifts toward longer wavelengths with increasing solar zenith distance, so that for $\Theta_0 = 84^\circ.3$ there is only one broad peak around 4,600 Å. An interesting feature is that the spectral distribution of the sky radiation changes very gradually with increasing solar zenith distance, showing a persistent preponderance in blue and near ultraviolet during most of the daylight hours.

In order to demonstrate the effect of reflection according to the Lambert law, the dash curve is included in Fig. 3, representing the spectrum of the sky radiation in the case of reflection with an albedo of 0.25, when the sun is at the zenith. It is seen that the contribution from such an assumed reflection law is quite large, and its effect is to accentuate the maxima and minima.

In all cases under the assumptions of pure Rayleigh scattering, there is very little sky radiation in red and infrared wavelengths. It should also be noted that there is considerable sky radiation in the near ultraviolet wavelengths not shown in Fig. 3. The computations could not be extended to wavelengths shorter than about 3,000 Å, due to the fact that the tables (SEKERA, BLANCH, 1952) of the functions γ_r and γ_r do not include values corresponding to $\tau > 1.00$.

4. Integrated radiations

It would be quite interesting to compare the theoretical results obtained by the method described above with the actual spectrum of sky radiation observed at the earth's surface on a very clear day. At least in the visible range, except perhaps for small differences around 6,000 Å (Chappuis region of ozone absorption), theory and observation should check quite well. Unfortunately, due perhaps to difficulties in instrument design, few or no observations exist providing a sufficiently detailed knowledge of monochromatic radiation from the whole sky. Observations so far available, including the more recent ones, generally have been concerned with sky and global radiation measured by bolometric methods, which means a summation of radiation fluxes of all wavelengths. Such data will allow only a very rough check of the theory, and must be compared with the *integrated* values of the sun, sky, and global radiation represented by the expressions (22). In the present case, the integration had to be limited to the range $0.29\ \mu \leqq \lambda \leqq 4.00\ \mu$, valid at sea level. This limitation may not effect the results appreciably since in the atmosphere below the main ozone layer, the solar energy at wavelengths shorter than 3,000 Å is almost completely cut off.

The results of the integration (by the trapezoidal rule) are given in Table VI, in watt per square meter, where the symbols H_0, $H_{.25}$ and $H_{.80}$ stand for the integrated absolute sky radiation for the case of no reflection, and reflection with albedo 0.25 and 0.80 respectively; G_0, $G_{.25}$ and $G_{.80}$ for the corresponding global radiation; and S for the integrated sun radiation. Referring to Fig. 2, the quantity S represents the areas under the lowermost curves, H_0 that of the hatched regions, and $H_{.25}$ that of the hatched plus black regions. For comparison one can use a *restricted solar constant* i.e. the area under the $\pi F_{0\lambda}$ curve of Fig. 2 in the range $.29\ \mu \leqq \lambda \leqq 4.00\ \mu$, computed by the same method, which amounts

Table V. The *absolute sky radiation* $\pi(F_0H)_\lambda$ in *watt per m² per 100 Å* on the same basis as Table IV

λ \ Θ_0	0°	53°.1	84°.3	λ \ Θ_0	0°	53°.1	84°.3
.29	1.20	0.78	0.10	.50	1.39	1.31	.076
.30	1.32	.90	0.12	.51	1.29	1.22	.73
.31	1.82	1.31	.18	.52	1.12	1.06	.66
.32	1.99	1.49	.22	.53	1.08	1.03	.66
.33	2.25	1.75	.28	.54	1.01	.96	.64
.34	2.06	1.65	.29	.55	.89	.86	.58
.35	2.02	1.65	.32	.56	.87	.83	.58
.36	1.92	1.61	.34	.57	.80	.77	.55
.37	1.77	1.52	.35	.58	.75	.72	.53
.38	1.61	1.40	.36	.59	.68	.66	.49
.39	1.58	1.39	.39	.60	.63	.60	.46
.40	2.27	2.04	.62	.65	.41	.40	.32
.41	2.38	2.14	.71	.70	.26	.26	.22
.42	2.22	2.01	.72	.75	.18	.18	.16
.43	1.91	1.74	.68	.80	.12	.12	.11
.44	2.04	1.87	.79	.85	.08	.08	.08
.45	2.06	1.91	.86	.90	.06	.06	.06
.46	1.98	1.83	.88	.95	.04	.04	.04
.47	1.81	1.69	.86	1.00	.02	.02	.02
.48	1.70	1.60	.83	1.50	.00	.00	.00
.49	1.46	1.38	.76				

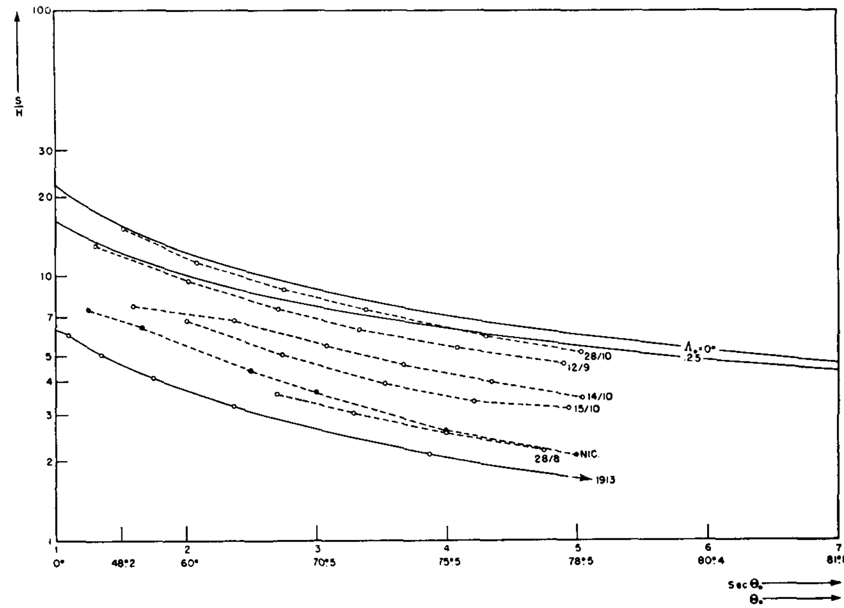

Fig. 4. Comparison of the ratio S/H of integrated sun and sky radiation (as a function of air mass or secΘ_0) with observed values. The full curves marked $A = 0$ and .25 represent the theoretical values for albedo zero and 0.25 computed in this paper (see text).

to 1,390.55 watt/m^2 or 1.99 cal/cm^2 min. Comparing this with G_0 at $\Theta_0 = 0°$, i.e. 1,330.7 watt/m^2, one can say that under these conditions, the sun and sky radiation received at the earth's surface from a pure Rayleigh atmosphere is about 95 % of the solar constant. In the extreme case of reflection with an albedo of 0.80, the integrated global radiation represents 101 % of the solar constant, an interesting result accounted for by the use of Lambert's law, already discussed earlier. The integrated global radiation drops off with increasing zenith distance, slowly at first, and more rapidly as Θ_0 increases. The dependance is such that log G_0 varies almost linearly with log μ_0.

Table VI. The *integrated absolute sun, sky and global radiation*, *S*, *H*, and *G* respectively, in *watt per m*^{2} over the range .29 $\mu \leq \lambda \leq$ 4.00, for four positions of the Sun. The subscripts 0, .25 and .80 refer to values corresponding to reflection with albedo 0.0, 0.25 and 0.80 respectively. The ratios *H/G* for different albedos are also shown for convenience. The solar constant in the range used is 1390.55 watt/m^2

Θ_0	0°	53°.1	84°.3	88°.8
H_0	57.0	50.9	23.8	7.8
$H_{.25}$	78.4	62.9	25.6	8.0
$H_{.80}$	134.7	94.5	28.9	8.6
S	1,273.7	728.6	86.7	10.1
G_0	1,330.7	779.5	110.5	17.9
$G_{.25}$	1,352.1	791.5	112.3	18.1
$G_{.80}$	1,408.4	823.1	115.6	18.7
H_0/G_0	0.0428	0.0653	0.2156	0.4377
$H_{.25}/G_{.25}$	0.0579	0.0795	0.2275	0.4440
$H_{.80}/G_{.80}$	0.0956	0.1148	0.2502	0.4595

From the values of Table VI, one can also appreciate the relative magnitude of the integrated sky and sun radiation, and how this varies with zenith distance. When the sun is at the zenith, H_0 is only 4 per cent while S is 96 per cent of G_0; but when $\Theta_0 = 84°.3$ the percentages are $21\frac{1}{2}$ and $78\frac{1}{2}$ respectively. Table VI also includes the ratios H/G.

A good set of early observations of the integrated sky radiation or 'sky brightness' were made by MOORE and ABBOT (1920) of the Smithsonian Institution in 1917, at Hump Mountain, North Carolina, 1462 meters above sea level. Values of net integrated fluxes of sun and sky radiation in Langleys as well as their ratio S/H, are given as a function of sun elevation. From an inspection of the values of Table VI it is seen that this ratio should drop with increasing solar zenith distance. Also from physical considerations, S/H should increase for all solar positions, if the normal optical thickness is decreased, i.e. if the observer is higher in the atmosphere. The nature of this ratio is seen better from the diagram in Fig. 4, in which the ordinates represent the ratio S/H (logarithmic scale) and the abscissae are marked in sec Θ_0 or the 'air mass'. The two top smooth curves marked $A = 0$ and $A = .25$ represent the theoretical variation of S/H_0 and $S/H_{.25}$ respectively. The five dash curves marked with the dates August 28, September 12, October 14, 15, and 28 result by merely drawing straight lines between the points corresponding to some of Moore and Abbot's observations (1920). In general the order of magnitude and slope of the observational data agree with the theory, however, most of the observed values are lower than the theoretical ones, contrary to expectations (i.e. considering the height of the observatory above sea level). One possible explanation would be that the atmosphere above Hump Mountain contained a large number of non Rayleigh scatterers during the days of observation.

This discrepancy with the theory appears even more clearly from the curve marked 1913, which represents two days' observations made during that year by the Smithsonian Institution at Mt. Wilson (cf. ABBOT ET AL., 1932, page 274), at an elevation of 1,737 meters. Since Mt. Wilson is higher than Hump Mountain, the corresponding values of S/H should be larger than at Hump Mountain. Note, however, that the opposite was observed. The explanation might be connected with the fact that the volcanic dust thrown into the upper atmosphere during the June 1912 eruption of Mt. Katmai in Alaska, was still visible in California in September 1913, as pointed out by the Smithsonian authors (cf. loc. cit., p. 268).

One more observation is shown by the curve marked "NIC", representing best conditions as observed by NICOLET and DOGNIAUX (1951)* at Uccle, Belgium, practically at sea level. This falls well below the theoretical curve for sea level, and furthermore the slope shows some disagreement with that of the theoretical curve. This might be connected with a systematic dependence of the turbidity of the atmosphere above Uccle, on the solar zenith distance. Note, however, that the volcanic dust condition at Mt. Wilson is more effective than the turbidity near sea level in reducing the values of S/H.

5. Discussion

Can one make any definite statements about the relation between the observed ratio S/H and the turbidity of the actual atmosphere? We do not believe one can, and disagree with MOORE and ABBOT's conclusions (1920), to the effect that turbidity or the existence of large particles in the atmosphere *necessarily* results in a lowering of the ratio S/H. This quantity in itself is not sufficient to tell us anything about the causes of deviation from the theory, since it represents the ratio of two integrals, both strongly dependent on the scattering mechanisms going on in the real atmosphere. Perhaps a more definite statement could be made with respect to volcanic dust in the high atmosphere: The opaqueness of this aerosol was demonstrated by the marked increase in normal optical thickness of the atmosphere in the years immediately following the Katmai eruption. Comparing the corresponding curve (marked 1913) and the theoretical one for zero albedo in Fig. 4, one notes that they are remarkably parallel, and since the vertical scale is logarithmic this indicates a constant factor of proportionality, about 0.30 in this case. If one assumes that the sky radiation was almost unaffected, this suggests that the volcanic dust acted as a *neutral* filter in the high atmosphere, absorbing radiation uniformly at all wavelengths. This of course is a highly tentative conclusion.

In general, a study of the ratio S/H as observed at different places shows remarkable agreement with the theory, considering the very restrictive assumptions made in the latter.

The quantitative results of this paper can be compared with the results obtained by F. BERNHARDT (1952, 1953). In his two papers, Bernhardt attempts to evaluate the effect of

* Nicolet furnishes values of the ratio H/G (cf. NICOLET, DOGNIAUX, 1951, p. 13) from which the S/H values can be deduced.

multiple scattering by the procedure suggested by F. LINKE (1942), and based on the assumptions (a) that the effect of higher order scattering is additive to that of the primary scattering, and (b) that the ratio of the increase in radiation due to the k-th order of scattering to the increase due to the $(k-1)$ th order of scattering, is constant. If $I^{(1)}$ denotes the specific intensity of diffuse sky radiation due to primary scattering, and $I^{(k)}$ its increase due to the k-th order scattering, then the specific intensity including the effect of all orders of scattering can be written, according to the above assumption, in the form

$$I = \sum_{k=1}^{\infty} I^{(k)} = I^{(1)} (1-q)^{-1},$$

where $q = \dfrac{I^{(2)}}{I^{(1)}} = \dfrac{I^{(3)}}{I^{(2)}} = \cdots \dfrac{I^{(k)}}{I^{(k-1)}} = \text{constant.}$

Now in the exact theory which was used in our discussion, the effects of the higher order scattering are included by means of successive iterations of four pairs of simultaneous, non-linear integral equations, defining the X- and Y-functions, mentioned under equation (15) above. The relation between successive orders of scattering is obviously not as simple as indicated in assumptions (a) and (b) above; this relation can be seen clearly from the values of successive approximations of the X- and Y-functions appearing under Table IX (SEKERA, BLANCH, 1952). Furthermore, in the computation of intensities due to the primary scattering, by HESS and LINKE (1942), and in the computation of the secondary scattering by BERNHARDT (1952, 1953), as well as in the earlier work of KING (1913), the polarization of the scattered light has been completely ignored and this leads to non-negligible differences as shown by CHANDRASEKHAR (1950, p. 264; cf. also VAN DE HULST [1949] p. 76, 77). Finally, the computations made by Hess and Linke, and as extended by Bernhardt, were performed with the use of mean values of the transmission coefficient $\bar{q}_\lambda$* and of the extraterrestrial flux πF_0 in the three spectral ranges: $\lambda < 0.4$, $0.4 \leqq \lambda < 0.6$, $0.6 \leqq \lambda < 0.9$.

* A quantity corresponding to $\exp(-\tau)$ in our notation.

It is thus quite difficult to interpret properly the differences between Bernhardt's and our results. In order to eliminate the differences resulting from the use of a different extraterrestrial flux, we had to deduce a mean relative sky radiation $\bar{H}_\lambda$ from our monochromatic values covering the above ranges, which was then multiplied by the corresponding extraterrestrial fluxes used by both Hess & Linke and Bernhardt. The resulting absolute fluxes of sky radiation are compared with Bernhardt's values in Fig. 5, which shows their variation with solar zenith distance Θ_0. The largest systematic difference appears in the curves marked 3, where Bernhardt's values exceed ours by about 60 per cent for all solar zenith distances shown. Now in this range the third order scattering is quite negligible, hence the corresponding values of the ratio $H_\lambda^{(1)}/H_\lambda$ appearing under section 3 above really represent the ratio $H_\lambda^{(1)}/(H_\lambda^{(1)} + H_\lambda^{(2)})$ from which the ratio $H_\lambda^{(2)}/(H_\lambda^{(1)} + H_\lambda^{(2)})$ can be easily deduced. This is about 0.11 for the beginning of the range ($\lambda = 0.809\,\mu$) and 0.04 for the end

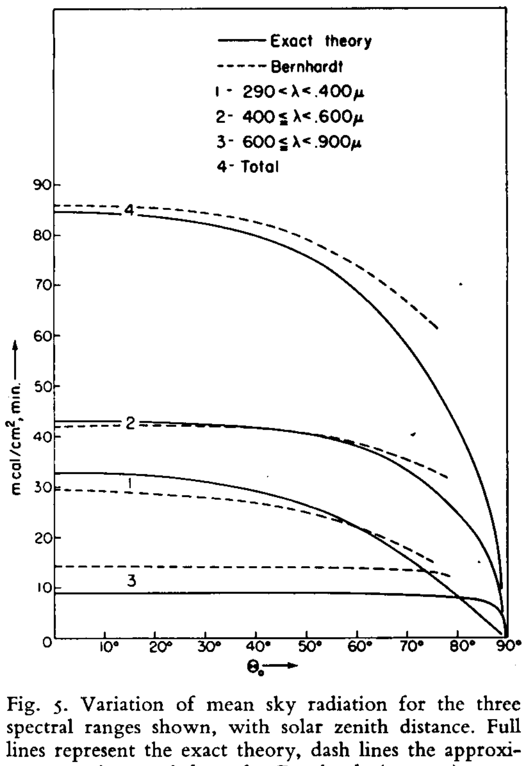

Fig. 5. Variation of mean sky radiation for the three spectral ranges shown, with solar zenith distance. Full lines represent the exact theory, dash lines the approximation carried out by Bernhardt (see text).

of the range ($\lambda = 0.618\ \mu$), with a mean value of 0.075, comparable with Bernhardt's value of 0.077. Similarly the transmission varies in the same range from 0.94 to 0.98, with a mean of 0.96 as compared with Bernhardt's value of 0.95. It is clear therefore that the difference in the curves 3 of Fig. 5 can only be accounted for by a difference in the respective sky radiation due to primary scattering.

This can be easily verified by evaluating mean values of $H_\lambda^{(1)}$ from our computations, for the same three spectral ranges, and comparing the absolute fluxes obtained with those given by Bernhardt as corresponding to primary scattering. Our values, obtained by graphical methods of finding the mean $H_\lambda^{(1)}$, are compared with those of Bernhardt below:

Spectral range	$\lambda < 0.4$	$0.4 \leq \lambda < 0.6$	$0.6 \leq \lambda < 0.9$
Absolute fluxes $H_\lambda^{(1)}$	15.33	32.91	8.48 mcal/cm² min
Bernhardt's values......	16.48	34.30	13.27 mcal/cm² min
Difference in %	7.5 %	4.2 %	56.5 %

The values shown in the last column above corroborate our statement that the large difference occuring in the curves 3 of Fig. 5 is due to a different estimate of the sky radiation from primary scattering. Since our values for $H_\lambda^{(1)}$ in the third range do not show any systematic deviation and are continuous when plotted next to the values of H_λ for shorter wavelengths (cf. Fig. 1); and since these $H_\lambda^{(1)}$ values approach asymptotically the theoretical value when $\tau \to 0$, we are inclined to believe that the values used by Bernhardt for the primary scattering are in error, at least in the third range.

Another discrepancy is evident if one considers that Bernhardt's values in the first two columns above also exceed ours somewhat, however the corresponding sky radiation due to all orders of scattering computed by Bernhardt's method is lower than ours for small solar zenith distance and exceeds our values for $\Theta_0 > 50°$, as seen from the curves marked 1 and 2 in Fig. 5. Thus it would seem that the evaluation of multiple scattering effects from those of lower order scattering by a simple assumption such as mentioned above, and the omission of the polarization of the scattered radiation lead to systematic deviations from the correct theoretical value: the method results in underestimating the sky radiation for high sun's elevations and in overestimating this radiation for low sun elevations.

Finally, with respect to actual measurements of monochromatic sky radiation, one of the rare works of this kind available is that carried out by Pettit (1932) in the late twenties. He was mainly interested in finding the ratio of ultraviolet to green direct solar radiation received at the ground, and in the determination of the solar extraterrestrial flux in the short wavelengths. From this work the only information which could be used here for comparison is his scanty measurements of sky radiation at .320 μ. For Pasadena, California (elevation 258 m.) the ratio H_λ/S_λ at this wavelength is given as 1.00 but unfortunately the exact solar altitude is not indicated, except that the sun was near the zenith. From our Tables IV and V, the ratio at sea level for this wavelength at $\Theta_0 = 0°$ turns out to be 1.99/2.91 or about 0.66. However, if one considers that this ratio increases rapidly with Θ_0, and should increase also with the increase of atmospheric turbidity, Pettit's value seems very reasonable.

Pettit also shows the variation of sky radiation at .320 μ with solar altitude (cf. Fig. 6 in Pettit, 1932) and finds a simple linear dependence. This does not quite check with the theory and the linear extrapolation to a zenith sun is not justified. The slope of the corresponding theoretical curve, if plotted on the same basis, would decrease gradually becoming horizontal for a zenith sun, as should be expected from physical considerations. The seemingly straight character of the plotted points might be due to the gradual increase in haze mentioned by the author, toward noon and early afternoon, which makes the afternoon plots fall above the forenoon ones.

A more complete and carefully made set of such observations giving the ratio H_λ/S_λ in narrow spectral bands would be highly desirable. It would be useful not only in checking the Rayleigh model for a pure atmosphere, but also in learning more about large particles existing in the real atmosphere, since the ratio H_λ/S_λ seems to be highly sensitive both to variations in wavelength and the size and character of the large particles present.

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