The Adjustment of a Non-balanced Velocity Field towards Geostrophic Equilibrium in a Stratified Fluid[1]

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Abstract

The previous work by ROSSBY and CAHN on the adjustment of a non-balanced velocity field towards geostrophic equilibrium has been generalized to incorporate the corresponding process in a stratified but incompressible fluid. It is shown that the character of the process essentially depends upon the width of the current, the velocity profile in the vertical and the vertical density gradient. As long as the vertical density gradient is small the adjustment of the mean motion of the fluid $\left(\int_0^{p_0} U dz\right)$ exactly corresponds to the one described by CAHN. On the other hand vertical variations of the unbalanced motion will set up internal oscillations which have considerable larger (vertical) amplitudes than is found for the adjustment of the mean motion. Furthermore, the speed of propagation of the gravity inertia waves thus generated (c_K) is considerably smaller than that of the waves generated during the adjustment of the mean motion (c_0). For values of the vertical density gradient as found in the oceans $c_K < 0.05\, c_0$. In the final state of equilibrium the mean motion of the fluid will be approximately the same as before the adjustment, while a considerable smoothing of the vertical gradients of the velocity field will take place, the more the broader the original current is.

1. Introduction

The motion of the atmosphere as well as the oceans takes place under approximate geostrophic balance. This has been known for a long time and has been used both in meteorology and oceanography to compute the state of motion more accurately than it can be observed directly. The geostrophic relation has been very useful as a *diagnostic* tool. The relation is merely an expression for the fact that usually the horizontal acceleration is one order of magnitude less than the Coriolis force and the horizontal pressure gradient. Thus it may be neglected in a first approximation. However, then it also follows that the geostrophic relation in itself cannot be used as a *prognostic* tool. Having realized this fact meteorologists for a long time have been studying the departures from geostrophic equilibrium to obtain a better understanding of the atmospheric processes and to derive relations that might be useful in forecasting. It must, however, be admitted that in most cases these attempts have not been very successful, probably to a large extent depending upon the difficulty of making the proper approximations in order to separate various physical processes from each other.

Lately CHARNEY and collaborators, however, (cf. CHARNEY 1949) have been able to develop useful methods for forecasting the upper flow patterns in the atmosphere by using the

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geostrophic approximation in the barotropic vorticity equation. This procedure essentially means a filtering out of the inertia oscillations that necessarily must develop for the adjustment of the wind field and mass field to each other in order to retain an approximate geostrophic equilibrium. One assumes that this adjustment is complete and takes place with an infinite speed and that the displacements that must occur are small in comparison to the large-scale motion of the atmosphere. The method has been quite successful and gives a first approximation of the development of the mean field of motion that even may be useful in practical forecasting. The realization of the physical sequence of events made it possible to apply the proper approximations at the right instant in the process of the development of the equations.

Undoubtedly there exist cases where it is not permissible to neglect this adjustment process towards geostrophic equilibrium to obtain acceptable results. This is probably still more important when trying to study the changes of the *vertical variation* of the wind field, where the stratification of the atmosphere plays an important role for the adjustment process. It is quite well-known, both from synoptic experience and from theoretical investigations (Rossby, 1938), that the adjustment of an unbalanced current in a stratified fluid towards geostrophic equilibrium involves a considerable redistribution of the mass field. We shall also see that the speed of propagation of the internal gravity-inertia waves, by which the adjustment takes place, is considerably smaller than the speed of the waves that develop during the adjustment in a homogeneous (barotropic) atmosphere. It is likely that some of the systematic errors that appear when forecasting with, for example, a two-parameter model of the atmosphere (Charney, Phillips, 1953) are due to the neglect of this adjustment process. To account for the apparence of such inertia-gravity waves in the process of numerical forecasting one may proceed in different ways. One may use the hydrodynamic equations without ever introducing the geostrophic approximation. This would probably increase the amount of computations for a 24-hour forecast by several magnitudes. It therefore seems very desirable to try to develop some simplified method in which the essential characteristics of the adjustment process have been considered. In order to be able to do this we must learn more about the adjustment of the flow and the pressure field to each other in idealized cases. We already now know some of the main features of this process essentially through the work by Rossby (1938), Cahn (1945) and Raethjen (1950). However, as yet no systematic treatment of a stratified atmosphere has been presented. Some principles have been derived by Rossby, but in particular a study of the *process* of adjustment in a fluid with a *continuous* density distribution is lacking.

This problem of maintaining an approximate geostrophic balance is also an important aspect of the dynamics of the ocean currents. The main difference from a hydrodynamic point of view between the oceans and the atmosphere is, of course, the facts that the sea-water may be considered as an incompressible fluid, the oceans have an upper free surface and the density difference between the top and the bottom is quite small. These characteristics simplify the mathematical treatment considerably and we shall therefore first discuss the adjustment towards geostrophic equilibrium in an *incompressible, stratified fluid of an approximately constant depth*, D. These restrictions will be removed in part II of this paper, which is going to appear in a later issue of this journal. It would, of course, be possible to derive the equations for the most general case first and then simplify the formulae obtained to apply to the more special problems. The procedure followed here is used primarily because the discussion of the upper boundary condition is believed to be more clear in this way, secondly because the method of approach is most easily followed in a relatively simple case.

Rossby in 1938 described the general features of the adjustment within an infinite current in a homogeneous incompressible fluid of uniform depth, initially not in geostrophic equilibrium. Let us assume that a certain amount of momentum has been communicated by some process (e.g. wind action) to an infinite strip parallell with the x-axis. However, no pressure gradients exist at this time. The momentum of the fluid is associated with a certain Coriolis force which will try to deflect the moving particles towards right (looking down-stream). Gradually fluid is accumulated to the right of the current while the free surface is lowered to the left of the current. In such a way the proper pressure gradient is built up and geostrophic equilibrium is approached. Rossby pointed out that certain inertia oscillations must be generated and a certain portion of the initial energy of the current is thus transformed into such oscillations. It was also shown how these considerations for the adjustment of a current in a single homogeneous layer may be generalized to the corresponding process in a fluid consisting of two or several homogeneous layers on top of each other but with different densities. The internal boundaries between the layers will be considerably more deformed than the free surface if the density differences between the various layers is small. In this case proportionally a larger portion of the kinetic energy of the initially unbalanced current is transformed into inertia-gravity oscillations.

Recently also RAETHJEN (1950) has published a study of the adaptation of the wind and pressure fields towards each other when initially not in balance. He points out that a current extending through the whole atmosphere will change very little during the adjustment, while a current only occupying a certain fraction of the atmosphere will be changed considerably more.

The papers by ROSSBY and RAETHJEN were mainly devoted to a study of the final state of equilibrium. No attempts were made to study the *process* of adjustment. This problem was attacked by CAHN (1945) who solved the problem for a single homogeneous layer of fluid. He demonstrated how the energy of the inertia oscillations very rapidly is carried away from the source region by damped travelling gravity waves the speed of which is $\sqrt{gD_0}$ (D_0 being the depth of the fluid). The final state of equilibrium is very closely reached already after a few oscillations.

The problem of how the adjustment of the velocity field towards geostrophic equilibrium takes place in a stratified fluid has remained unsolved up to the present time. It is the purpose this paper to discuss this problem in some detail. In order to describe the physical process most clearly we shall first treat a case similar to the one discussed by ROSSBY and CAHN (l.c.).

2. The basic differential equations

Let us consider an incompressible fluid of the average depth, D_0, which has an infinite extent in the horizontal directions x and y. A certain vertical density gradient exists in the fluid. We assume that the density is given by

$$\varrho = \varrho_0 (1 - \alpha\sigma) \tag{1}$$

where σ is a function of the vertical coordinate (z) being zero at the bottom and unity at the upper free surface. Thus, both the bottom and the free surface are assumed to be isosteric surfaces. It will furthermore be assumed that the vertical density gradient is small ($\alpha \ll 1$).

We shall consider an infinite current, u_0 along the x-axis which is assumed to be independent of x ($\partial/\partial x \equiv 0$). Thus it is sufficient to study what happens in a plane perpendicular to this axis (yz-plane). Let ε and ζ denote the horizontal and vertical displacements in this plane. Through making use of the assumption of incompressibility we obtain the following continuity equation

$$\frac{\partial \varepsilon}{\partial y} + \frac{\partial \zeta}{\partial z} = 0. \tag{2}$$

We want to use σ as our independent vertical coordinate and transform (2) accordingly. We may write

$$\frac{\partial z}{\partial \sigma} = \left(\frac{\partial z}{\partial \sigma}\right)_0 + \frac{\partial \zeta}{\partial \sigma} = D_0 + \frac{\partial \zeta}{\partial \sigma} \tag{3}$$

$\partial z/\partial \sigma$ is an expression for the vertical stability of the fluid and $(\partial z/\partial \sigma)_0 = D_0$ is the value of $\partial z/\partial \sigma$ in an undisturbed state. We have here assumed that σ is a linear function of z. Now the assumption of small perturbations means

$$\frac{\partial \zeta}{\partial \sigma} \ll D_0 \tag{4}$$

and if $\mu = \zeta/D_0$ we may write (2) as

$$\frac{\partial \varepsilon}{\partial y} + \frac{\partial \mu}{\partial \sigma} = 0. \tag{5}$$

The two horizontal components of the equation of motion are

$$\frac{du}{dt} = fv \tag{6}$$

$$\frac{dv}{dt} = -fu - \frac{1}{\varrho}\frac{\partial p}{\partial y} \tag{7}$$

where already the assumption of $\partial/\partial x \equiv 0$ has been introduced. f is the Coriolis parameter and is assumed to be constant. u and v are the horizontal components of velocity along the x and y-axes and p denotes pressure. Neglecting higher order terms we obtain by integration of equation (6)

$$u = u_0 + \left(f - \frac{\partial u_0}{\partial y}\right)\varepsilon.$$

We shall *a priori* exclude the possibility for dynamic instability to occur and assume $\partial u_0/\partial y \ll f$. It is then seen from equation (7) that any geostrophically balanced field of motion may be added without changing the character of the adjustment process. We may then simply assume that no horizontal pressure gradients exist initially i.e. the density surfaces as well as the free surface are horizontal. Thus u_0 denotes the initially unbalanced field of motion along the x-axis and

$$u = u_0 + f\varepsilon. \tag{8}$$

In equation (7) we must transform $\partial p/\partial y$ into an expression only containing our depending variables defined previously. By making use of the hydrostatic relation and assuming that no external forces act on the free surface we obtain

$$p = g\int_z^D \varrho dz \tag{9}$$

where D is the actual depth of the fluid and g acceleration of gravity. Through integration by parts and making use of (1) in differentiated form we get

$$p = g\,(\varrho_s D - \varrho z) + \alpha\varrho_0 g \int_\sigma^1 z d\sigma \tag{10}$$

ϱ_s denotes the density at the free surface and z is the height of the density surfaces (function of σ). Now

$$\left(\frac{\partial p}{\partial y}\right)_\sigma = \left(\frac{\partial p}{\partial z}\right)_z + \frac{\partial p}{\partial z}\cdot\frac{\partial z}{\partial y} = \left(\frac{\partial p}{\partial y}\right)_z - g\varrho\frac{\partial z}{\partial y} \tag{11}$$

Differentiation of (10) with respect to y, keeping σ constant, gives

$$\left(\frac{\partial p}{\partial y}\right)_\sigma = g\varrho_s\frac{\partial D}{\partial y} - g\varrho\frac{\partial z}{\partial y} + \alpha\varrho_0 g\int_\sigma^1 \frac{\partial z}{\partial y}d\sigma. \tag{12}$$

Combination of (11) and (12) yields

$$\left(\frac{\partial p}{\partial y}\right)_\sigma = g\varrho_s\frac{\partial D}{\partial y} + \alpha\varrho_0 g\int_\sigma^1 \frac{dz}{\partial y}d\sigma. \tag{13}$$

Now

$$z = z_0 + \zeta = z_0 + \mu D_0 \tag{14}$$

where z_0 is the height of the σ-surfaces in the undisturbed state. Thus

$$\frac{\partial z}{\partial y} = D_0\frac{\partial \mu}{\partial y}. \tag{15}$$

Assuming small perturbations allows us to neglect the convective accelerations in dv/dt. Furthermore, $\alpha \ll 1$ and thus ϱ_s may be approximated by ϱ_0 in (13). With the aid of (13) and (15) we may thus transform (7) into

$$\frac{\partial v}{\partial t} = -fu - g\frac{\partial D}{\partial y} - \alpha g D_0\int_\sigma^1 \frac{\partial \mu}{\partial y}d\sigma. \tag{16}$$

The three equations (5), (8) and (16) and the definition of v

$$v = \frac{\partial \varepsilon}{\partial t} \tag{17}$$

represent a complete system of equations for the four dependent variables ε, μ, u and v as functions of y, σ and t. We observe that $D = D_0\,(1 + \mu_s)$, index "s" indicating values at the free surface of the fluid. We shall next eliminate three of those variables in order to obtain a single differential equation with only one dependent variable. Because of the character of the boundary conditions it is most suitable to retain μ. It is easily verified that the following differential equation is obtained

$$\frac{\partial^4\mu}{\partial t^2\partial\sigma^2} = f\frac{\partial^2 u_0}{\partial\sigma\partial y} - f^2\frac{\partial^2\mu}{\partial\sigma^2} - \alpha g D_0\frac{\partial^2\mu}{\partial y^2}. \tag{18}$$

If we introduce the vertical velocity as a new variable defined as

$$W = 1/f\frac{\partial\mu}{\partial t} \tag{19}$$

we get

$$\frac{\partial^4 W}{\partial t^2\partial\sigma^2} = -f^2\frac{\partial^2 W}{\partial\sigma^2} - \alpha g D_0\frac{\partial^2 W}{\partial y^2}. \tag{20}$$

Exactly the same equation is satisfied by v.

It is suitable to introduce non-dimensional variables here. In analogy with Rossby we may define a "radius of deformation" $R = \sqrt{\alpha g D_0} \cdot f^{-1}$. As unit of time we choose f^{-1}. The new non-dimensional variables are then defined by the following relations

$$\begin{cases} \varepsilon = R\varepsilon' \\ y = R\eta \\ t = \tau \cdot f^{-1} \\ v = R \cdot f \cdot V \end{cases} \tag{21}$$

μ, w and σ are already non-dimensional. Equation (20) now becomes

$$\boxed{\frac{\partial^4 W}{\partial \sigma^2 \partial \tau^2} + \frac{\partial^2 W}{\partial \sigma^2} + \frac{\partial^2 W}{\partial \eta^2} = 0} \tag{22}$$

We also shall need a relation between W and V which is obtained from (5) by differentiation with respect to time

$$\frac{\partial V}{\partial \eta} + \frac{\partial W}{\partial \sigma} = 0. \tag{23}$$

3. Adjustment within a single stratified layer of fluid

a. Boundary and initial conditions

The *bottom of the fluid* remains fixed at all times and thus

$$\sigma = 0: \ W \equiv 0. \tag{24}$$

At the free surface we get by putting $\sigma = 1$ in equation (16) and introducing the expression for D

$$\sigma = 1: \quad \frac{\partial v}{\partial t} = -fu - gD_0 \frac{\partial \mu}{\partial y}. \tag{25}$$

This equation is transformed into non-dimensional form. In order to eliminate U as a dependent variable we differentiate with respect to τ and make use of (8) and (17). Thus we get

$$\sigma = 1: \quad \frac{\partial^2 V}{\partial \tau^2} = -V - \frac{1}{\alpha} \frac{\partial w}{\partial \eta} \tag{26}$$

which will be the boundary condition at the free surface.

The only lateral boundary conditions that may be applied are the requirements that W and V remain finite when $\eta \to \infty$.

We shall investigate the adjustment towards geostrophic equilibrium of an arbitrary current within this stratified fluid, which initially is not in geostrophic balance. Let us assume that no motion exists initially in the plane perpendicular to the current

$$\tau = 0: \ V = W = 0. \tag{27}$$

However, if geostrophic balance does not exist, we find from equation (7) that $\partial v/\partial \tau \neq 0$. Thus the fluid will acquire a velocity in the η-direction and vertical motions will also appear as soon as $\partial(\partial V/\partial \tau/\partial \eta \neq 0$. It is obvious from equation (7) that it will be sufficient to know the unbalanced component of the motion in the x-direction to be able to determine the accelerations completely and thus also the following development. Thus, as was mentioned before, it is sufficient to study a case where no horizontal pressure gradients exist initially, since any geostrophically balanced motion may be superimposed upon the solution obtained. If U_0 is the value of this non-balanced component of the motion in non-dimensional form we obtain

$$\tau = 0: \quad \frac{\partial V}{\partial \tau} = -U_0 \tag{28}$$

where $$u_0 = R \cdot f \cdot U_0 \tag{29}$$

It is necessary to transform equation (28) into one containing W instead of V for reasons that will appear later. Differentiation with respect to η gives

$$\frac{\partial}{\partial \tau}\left(\frac{\partial V}{\partial \eta}\right) = -\frac{\partial U_0}{\partial y} = -\frac{\partial}{\partial \tau}\left(\frac{\partial W}{\partial \sigma}\right) \tag{30}$$

where we have made use of (23). We know that $W \equiv 0$ at $\sigma = 0$ and thus by integration with respect to σ we obtain

$$\tau = 0: \quad \frac{\partial W}{\partial \tau} = \int_0^\sigma \frac{\partial U_0}{\partial \eta} d\sigma. \tag{31}$$

This will be the second initial condition to be used together with (27).

b. Solution of the differential equation

Let us assume that W may be written

$$W = \sum_K A_K(\eta, \tau) \sin \lambda_K \sigma \tag{32}$$

by which assumption the boundary condition (24) is automatically fulfilled. Since the fluid has a finite extension in the vertical it is reasonable to assume a representation in the form of a series, *i.e.* only certain values of λ_K (eigen values) may be chosen in order to satisfy both the differential equation and the boundary conditions at the upper and lower boundaries. We shall later see that this assumption is correct and that λ_K is determined in such a way that the functions, $\sin \lambda_K\sigma$, fulfil the necessary orthogonal requirements. Thus any continous function W may be expressed in the form of a series as the one given in (32). (The reader is referred to, for example, RIEMANN-WEBER, 1918, COURANT-HILBERT, 1931, for a more detailed discussion of these and other problems arising in connection with the solution of partial differential equations.) Introducing (32) into the differential equation (22) we obtain

$$\sum_K \left[-\lambda_K^2 \frac{\partial^2 A_K}{\partial \tau^2} - \lambda_K^2 A_K + \frac{\partial^2 A_K}{\partial \eta^2}\right] \times \times \sin \lambda\sigma_K = 0. \qquad (33)$$

This relation must be valid for all values of σ. It follows that

$$\frac{\partial^2 A_K}{\partial \tau^2} - \frac{1}{\lambda_K^2}\frac{\partial^2 A_K}{\partial \eta^2} + A_K = 0 \qquad (34)$$

as the summation over $\sin \lambda_K\sigma$ is a unique and complete representation. This differential equation is hyperbolic and is most easily solved by Riemann's method provided the initial conditions may be properly expressed in terms of A_K.

Next we must satisfy the boundary condition at the free surface. From (23) we obtain

$$\frac{\partial V}{\partial \eta} = -\frac{\partial W}{\partial \sigma} = -\sum_K \lambda_K A_K \cos \lambda_K\sigma. \qquad (35)$$

In order to express all terms in (26) in series with respect to A_K we must differentiate this boundary condition with respect to η. Introduction of the expressions for W and $\partial V/\partial \eta$ then yields

$$-\sum_K \lambda_K \left[\frac{\partial^2 A_K}{\partial \tau^2} + A_K\right] \cos \lambda_K + + \frac{1}{\alpha}\sum_K \frac{\partial^2 A_K}{\partial \eta^2} \sin \lambda_K = 0. \qquad (36)$$

Making use of the basic differential equation (34) we obtain

$$\sum \left[\frac{1}{\lambda_K}\cos \lambda_K - \frac{1}{\alpha}\sin \lambda_K\right]\frac{\partial^2 A_K}{\partial \eta^2} = 0. \qquad (37)$$

Thus the upper boundary condition is satisfied provided

$$\boxed{\operatorname{tg} \lambda_K = \frac{\alpha}{\lambda_K}}\,. \qquad (38)$$

λ_K may be chosen in this way, since the functions $\sin \lambda_K\sigma$ then fulfil the orthogonal requirements mentioned above (COURANT-HILBERT, 1931).

Thus having satisfied both the upper and lower boundary conditions, the initial conditions given by (27) and (31) have to be expressed in terms of A_K. Equation (27) gives

$$\sum_K A_K \sin \lambda_K\sigma = 0. \qquad (39)$$

This should be valid for all values of σ and thus

$$\tau = 0: \quad A_K = 0. \qquad (40)$$

The second initial requirement (31) gives

$$\sum_K \frac{\partial A_K}{\partial \tau} \sin \lambda_K\sigma = \int_0^\sigma \frac{\partial U_0}{\partial \eta} d\sigma. \qquad (41)$$

The right hand side of (41) may be expanded in a series of the same type as the one applied to W. We may write

$$g(\eta,\sigma) = \int_0^\sigma \frac{\partial U_0}{\partial \eta} d\sigma = \sum_K B_K \sin \lambda_K\sigma \qquad (42)$$

or

$$\frac{\partial U_0}{\partial \eta} = \sum_K B_K \lambda_K \cos \lambda_K\sigma. \qquad (43)$$

It follows from the boundary condition that $g(\eta,0) \equiv 0$ and thus the representation in (42) is unique and complete. It is possible to express the coefficients B_K in terms of $\partial U_0/\partial \eta$. We finally get

$$B_K = \frac{1}{\lambda_K}\,\frac{2\int_0^1 g(\eta,\sigma)\cos \lambda_K\sigma\, d\sigma}{1 + \frac{1}{2\lambda_K}\sin 2\lambda_K}. \qquad (44)$$

By combination of (41) and (42) we obtain

$$\tau = 0: \ \frac{\partial A_K}{\partial \tau} = B_K \tag{45}$$

as the second initial condition to the hyperbolic equation (34) where B_K is determined by (44). We now get as the complete solution of (34)

$$A_K(\eta, \tau) = -\frac{1}{2}\int_{-\tau}^{\tau} B_K\left(\eta - \frac{s}{\lambda_K}\right)\cdot \cdot J_0(\sqrt{\tau_2 - s_2})\, ds. \tag{46}$$

Here J_0 denotes the zero order Bessel function.

Thus W may be computed from (32) and V is then obtained from (35) by integration with respect to η ($V = 0$ for $\eta = \infty$):

$$V = -\int_{-\infty}^{\eta} \sum_K \lambda_K A_K \cos \lambda_K \sigma d\eta. \tag{47}$$

The displacements of the particles at every moment are obtained from (17) and (19)

$$\begin{cases} \varepsilon' = \int_0^{\tau} V d\tau \\ \mu = \int_0^{\tau} W d\tau. \end{cases} \tag{48}$$

c. Final state of equilibrium

It seems not possible to evaluate the two integrals (48) exactly when $\tau \to \infty$. We may, however, obtain the final state of equilibrium directly from the differential equation (18) by putting $\partial/\partial\tau = 0$. Let us also introduce non-dimensional quantities and we obtain

$$\frac{\partial^2 \mu}{\partial \sigma^2} + \frac{\partial^2 \mu}{\partial \eta^2} = \frac{\partial^2 U_0}{\partial \sigma \partial \eta}. \tag{49}$$

The boundary condition at the bottom of the fluid is as before

$$\sigma = 0: \quad \mu = 0. \tag{50}$$

Putting $\partial v/\partial t = 0$ in equation (25) and introducing the expression for u given by (8) gives us the following relation at the free surface ($\sigma = 1$):

$$f u_0 + f^2 \varepsilon = -g D_0 \frac{\partial \mu}{\partial y} \tag{51}$$

or in non-dimensional form

$$\sigma = 1: \ U_0 + \varepsilon' = -\frac{1}{\alpha}\frac{\partial \mu}{\partial \eta}. \tag{52}$$

Let us assume that the solution of equation (49) may be expressed in the form

$$\mu = \sum_K P_K(\eta) \sin \lambda_K \sigma. \tag{53}$$

Thus the lower boundary condition (50) is satisfied. Substituting (53) and (43) into (49) gives

$$\sum_K \left[\frac{d^2 P_K}{d\eta^2} - \lambda_K^2 (P_K - B_K)\right] \sin \lambda\, \sigma = 0. \tag{54}$$

Since this relation should be valid at all levels we get

$$\frac{d^2 P_K}{d\eta^2} - \lambda_K^2 P_K = -\lambda_K^2 B_K. \tag{55}$$

In the same way as before the boundary condition (52) at $\sigma = 1$ gives us a relation for the determination of λ_K, which becomes identical with (38).

In the following applications we shall consider a distribution of U_0 which is symmetrical around $\eta = 0$. Thus B_K is an antisymmetric function and the same will be true for P_K. We may then also assume that $P_K = 0$ for $\eta = 0$. If the initial current is limited to a region $|\eta| \leq a$, $B_K = 0$ for $|\eta| \geq a$. Hence we obtain from (55) that P_K will approach zero when $\eta \to \infty$ as $\exp(-\lambda_K \eta)$, which is in agreement with the results obtained earlier by ROSSBY (l.c.).

Equation (55) is most easily solved numerically. For small values of K and a, $P_K \ll B_K$ and we may neglect P_K for $|\eta| \leq a$ which simplifies the integration for this region. For large values of K, λ_K becomes large and we may immediately write down the approximate solution $P_K = B_K$.

To obtain the final velocity distribution we transform equation (8) into non-dimensional form

$$U = U_0 + \varepsilon'. \tag{56}$$

From the continuity equation (5) we get

$$\varepsilon' = -\sum_K \lambda_K \cos \lambda_K \sigma \int_{-\infty}^{\eta} P_K\, d\eta \tag{57}$$

and from (43)

$$U_0 = \sum_K \lambda_K \cos \lambda_K \sigma \int_{-\infty}^{\eta} B_K \, d\eta. \tag{58}$$

These formulae will be discussed in the following section.

d. Discussion of the results

First some general comments will be given, then a description of the adjustment process in a few specific cases will be presented. These examples have been chosen to be of interest both to oceanographers and meteorologists. However, the application to the atmosphere should be made with caution, in particular since atmospheric conditions may be treated in a more exact way as will be shown in a forthcoming paper.

By and large the adjustment of the initially unbalanced current will follow the scheme outlined by Rossby (1938). However, it is seen from the preceeding sections that the detailed character of this process is determined by *the velocity profile of the basic flow* along the vertical. Each value of K corresponds to a certain periodic (sinusoidal) distribution of the basic motion with respect to σ in such a way that the basic velocity profile is practically independent of height for $K = 0$, it is zero at one intermediate level for $K = 1$, at two levels for $K = 2$, etc. (cf. Fig. 1). Any given velocity distribution may be decomposed in such a series of components (modes). Each one of these modes will give rise to certain

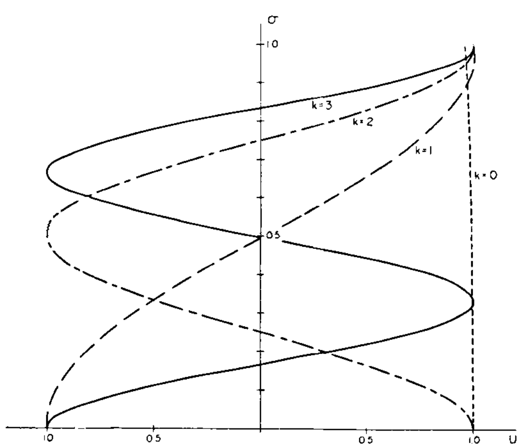


Fig. 1. The distribution of the basic velocity with depth which gives rise to oscillations of zero, first, second and third mode ($K = 0, 1, 2, 3$).

adjustment oscillations, which travel from the region of the original, unbalanced current in the direction of both the positive and negative y-axis. The amplitudes of the waves thus generated decrease with time and the motion gradually approaches an equilibrium state. The speed of the waves depends upon the value of K. Let us for the time being consider a point disturbance, e.g. $B_K \neq 0$ only in a small region around the origin, $\eta = 0$. It is then easily seen from (46) that $W = V = 0$ as long as $\tau < \lambda_K |\eta|$. Thus the influence will propagate with a maximum velocity C_K which is given by

$$C_K = \frac{1}{\lambda_K}. \tag{59}$$

Since α is a small quantity we get the following approximate expressions for λ_K from equation (38)

$$\begin{cases} \lambda_0 = \sqrt{\alpha} \\ \lambda_K = K\pi + \dfrac{\alpha}{K\pi}, \; K > 0. \end{cases} \tag{60}$$

Thus C_K becomes (expressed in ordinary units and then denoted by c_K)

$$\begin{cases} c_0 = \sqrt{gD_0} \\ c_K = \dfrac{\sqrt{\alpha}}{K\pi} c_0, \; K > 0. \end{cases} \tag{61}$$

Hence, c_0 *is independent of the vertical stratification of the fluid as long as the total density difference is small and it has the same value as was found by* Cahn (1945) *to be the speed of propagation of the waves generated during the adjustment of a homogeneous fluid.* It is seen from (43) that the basic current should be approximately the same at all levels in order only to generate this kind of waves (cf. also fig. 1, $K = 0$). For a value of $\alpha = 2 \cdot 10^{-3}$, which corresponds to conditions in the ocean, the speed at the surface should be 99.9 % of the speed at the bottom of the fluid. Thus $K = 0$ *describes the adjustment of the mean motion of the fluid.*

As soon as vertical shear exists in the current, waves will be generated with one or several nodal surfaces. The speed of propagation of the waves thus generated will be considerable smaller than that of the waves of zero mode. For the value of α chosen above we obtain

$$\begin{cases} c_1 = 0.014\ c_0 \\ c_K = \dfrac{c_1}{K}. \end{cases} \tag{62}$$

The speed of these waves will only be a few per cent or less of the speed of the waves that occur during the adjustment of the zero mode or barotropic component of the motion.

The amplitude of the waves depends upon the horizontal velocity profile and the mode. Equation (43) shows that the horizontal shear (for the particular mode studied) is proportional to $\lambda_K B_K$. If the current is very narrow and we consider some of the lower modes, where λ_K is small, it means that $B_K \neq 0$ for only a small interval of s in equation (46). Over this interval J_0 $(\sqrt{\tau^2 - s^2})$ varies only slightly. Let s_0 denote the distance from the point we are considering to the centre of the current and let $2a$ be the width of the current. With the assumption that $a \ll s_0 \ll \tau$ we may transform (46) into

$$A_K = \frac{s_0 J_0'(\tau)}{2\tau} \int_{-\tau}^{\tau} U_{0K}\left(\eta - \frac{s}{\lambda_K}\right) ds. \tag{63}$$

Here U_{0K} denotes the basic velocity profile corresponding to B_K. The relation (63) shows the decrease in amplitude with time as well as the dependency upon the width and strength of the original unbalanced current.

We have assumed small perturbations in the derivation of the formulae above. It is quite obvious that the vertical amplitude has to be small compared with D_0 for $K=0$; for $K>0$ it must be small compared with D_0/K. To secure that this is the case the unbalanced current must not be too strong and the limit is lower the higher the mode is with which we are concerned. In reality strong geostrophically unbalanced motions seldom exist at least not for the lower modes. The physical reason behind this upper bound of the momentum is easily found. A certain mean field of motion (zero mode) must be balanced by a slope of the free surface of the fluid. If the current be extremely broad and strong the difference in depth on both sides of the current becomes comparable with the total depth itself in which case it is not permissible to linearize the equations in the way we have done. However, such a case is probably of little practical interest. Conditions become quite different if we have vertical velocity gradients. The slopes of the isosteric surfaces *within* the fluid become appreciable already for a moderate increase of velocity with height, in particular for small values of α or large values of f. For example let us assume $\alpha = 2 \cdot 10^{-3}$ and an average increase of the velocity of 2 m/sec per km. The slope of the internal density surfaces must then be about 5 m/km in order to obtain geostrophic balance ($f = 10^{-4}\,\text{sec}^{-1}$). If the depth of the fluid is 1,000 m, geostrophic equilibrium cannot be established for a current of this type broader than about 200 km. For the higher modes this effect becomes still more important, since the slope of the density surfaces must change sign for a change of σ given by $\triangle\sigma = 1/K$. Our initial assumption, that the free surface and the bottom of the fluid coincide with isosteric surfaces cannot be maintained any longer. This example shows clearly that the stratification or *static stability* and the rotation of the earth causing a *dynamic stability* are the two important factors that determine the general character of the motions within the fluid. These problems have lately been discussed by ELIASSEN (1952).

We shall next study the changes of the velocity distribution obtained in the final state of equilibrium. It was already shown by ROSSBY (1938) that for a homogeneous current the changes largely depend upon the width of the current in such a way that they are less the more narrow the current is. It is seen from equation (55) that if $\eta\lambda_K = \xi$ we obtain

$$\frac{d^2 P_K}{d\xi^2} - P_K = -B_K. \tag{64}$$

Thus λ_K is merely a scale factor. If B_K is the same function of ξ for the different modes the final state of equilibrium P_K will be the same, expressed as function of the same variable ξ. The character of this state of equilibrium will mainly depend upon the width of the original unbalanced current in a way that is very similar to the results obtained by ROSSBY (l.c.). Figure 2 shows how an original velocity profile, U_0 (solid line), is modified during the process of adjustment. A norrow current will remain practically unchanged while a broad current will decrease considerably in intensity and counter-currents will develop on both sides of it. We also see that the different modes

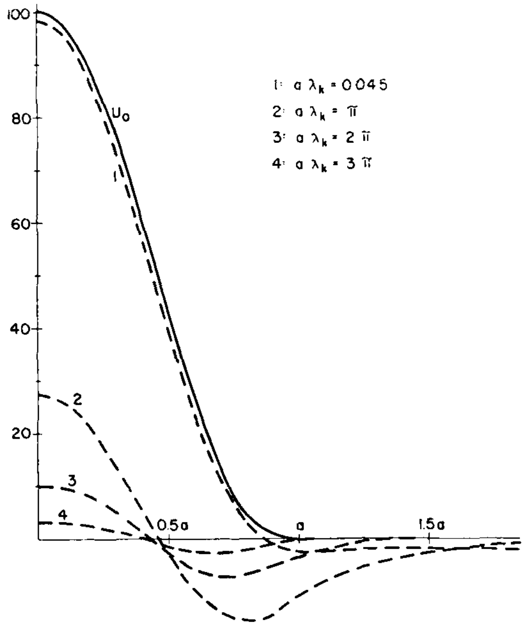


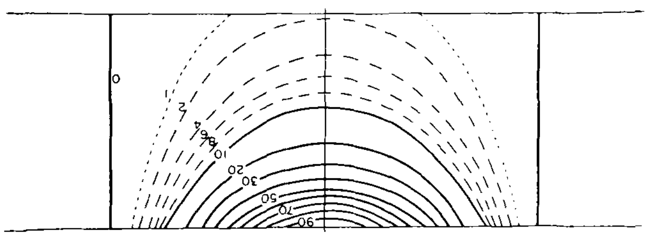
Fig. 2. The modification of a given horizontal velocity profile (U_0, solid line) as a result of the adjustment towards geostrophic equilibrium. a is the halfwidth of the current in the unit: $\sqrt{\alpha g D_0} \cdot f^{-1}$ ($\sim$45 km). For a given mode (a given value of λ_K from eq.(60)) the various dashed curves show the dependency of the final adjustment upon the width of the current. For the value $a = 1$ the curves show the final horizontal velocity profiles for the four lowest modes.

will be affected in different ways when the width of the current is given. Thus the higher modes will be considerably more reduced than the lower ones. A larger portion of the initial kinetic energy will be transformed into oscillatory energy of the system and is dispersed into the surrounding fluid masses. The final velocity profile in the surroundings is exactly the one found by Rossby as is easily seen from equation (55). Since $B_K = 0$ in the surroundings the solution will be proportional to $\exp\,(-\lambda_K|\eta|)$.

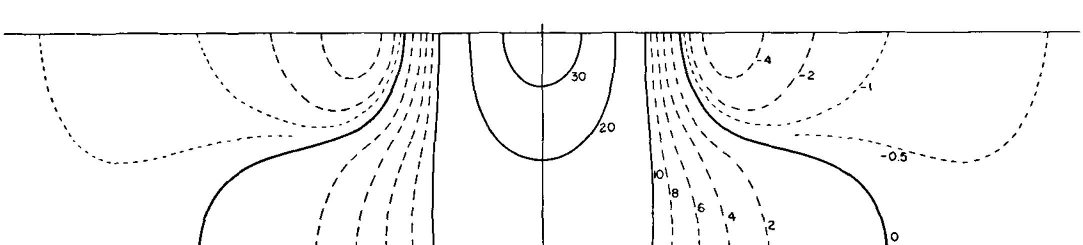
Since the higher modes always are reduced more than the lower ones the adjustment of an arbitrary unbalanced current towards geostrophic equilibrium always means a smoothing of the original vertical velocity profile. It may be of some interest to study such a case somewhat closer. As before we choose $\alpha = 2 \cdot 10^{-3}$, $D_0 = 1{,}000$ m and $f = 10^{-4}$ sec^{-1}; thus $R = 45$ km. Let us assume that the original unbalanced velocity distribution is defined by

$$\frac{\partial U_0}{\partial \eta} = -\,U_0^{\star}\, e^{\sigma^{-1} H}\left[\left(\frac{\eta}{a}\right)^5 - 2\left(\frac{\eta}{a}\right)^3 + \frac{\eta}{a}\right]. \quad (65)$$

This velocity field is given in figure 3 a for the particular case when $H = 0.25$ and $a = 1$ (*e.g.* the width of the current is 90 km). The horizontal velocity distribution is the one



3 a.



3 b.

Fig. 3. The adjustment of a given current to geostrophic equilibrium. The upper figure shows the unbalanced basic current, at $t = 0$, the lower figure gives the final state of equilibrium ($t \rightarrow \infty$).

corresponding to the solid curve in figure 2 and the vertical velocity profiles at different distance from the centre of the current are shown as solid lines in figure 4. For the width here chosen all higher modes ($K > 3$) will disappear almost completely in the final state of equilibrium and the lower ones are reduced considerably except the zero mode. The percentage reduction is shown in figure 2, where the values of $\lambda_K a$ have been chosen to illustrate the changes for the various modes in this particular case. The final velocity distribution is given in figure 3 b and the vertical velocity profiles are shown as dashed curves in figure 4. The changes of the horizontal velocity profiles are given in figure 5. We notice the following facts:

1) The main current has become more uniform in the centre.
2) The average speed of the current ($\int_0^1 U d\sigma$) has remained approximately unchanged.
3) The counter-currents have their strongest intensity at the boundary of the original current, and the vertical velocity gradients are large compared with the intensity of the counter-currents themselves.
4) The counter-currents become more uniform the further away from the centre of the original current we proceed.

We shall finally give an illustration of how the final equilibrium is gradually approached. Since the amount of computations is large, we shall here restrict ourselves to a description of the adjustment of the first mode. The changes of the other modes are very similar. Figures 6 shows the shape of the density

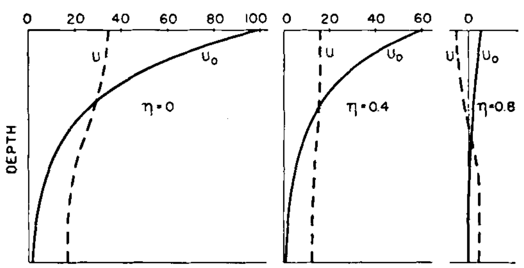


Fig. 4. Vertical profiles of the basic current before solid lines) and after the adjustment (dashed lines) as a function of the distance from the centre of the current. The figure is based on the velocity distibution in figure 3.

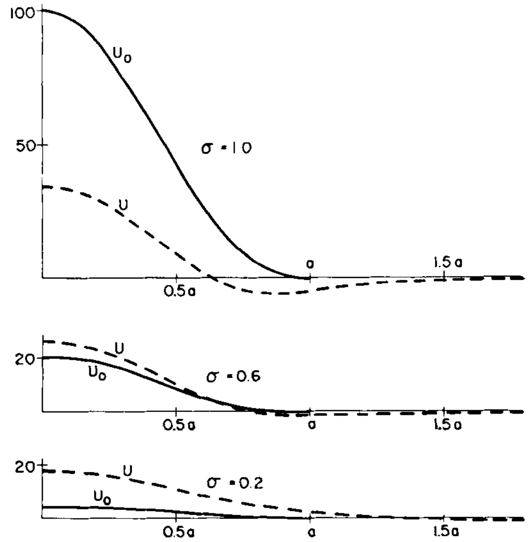


Fig. 5. Horizontal velocity profiles at different heights above the bottom before and after the adjustment. The curves represent horizontal sections in the case given in figure 3.

surfaces, originally horizontal, at three different instances after the beginning of the adjustment *viz.* after $6^h\ 5^m$, $15^h\ 30^m$ and 25^h. The originally unbalanced current was restricted to the region inside the two vertical solid lines and the original horizontal velocity profile was given by the solid curve in Fig. 7. As before $a = 1$. In order to see the changes more clearly the deformations of the density surfaces have been made larger than is permissible according to the linearized theory. The final state of equilibrium is shown as dashed lines.

The discussion above has been carried out starting with an unbalanced velocity field. Of course it applies as well to a mass field initially not in geostrophic balance. The discussion becomes quite analogus since we always may define the geostrophic velocity corresponding to the actual mass distribution and thus also define an unbalanced velocity field. It is then clear that the higher modes of the mass field distribution will change very little while the lower modes are practically completely adjusted towards the existing geostrophic wind field.

At this stage of the work it would be premature to discuss the implications that

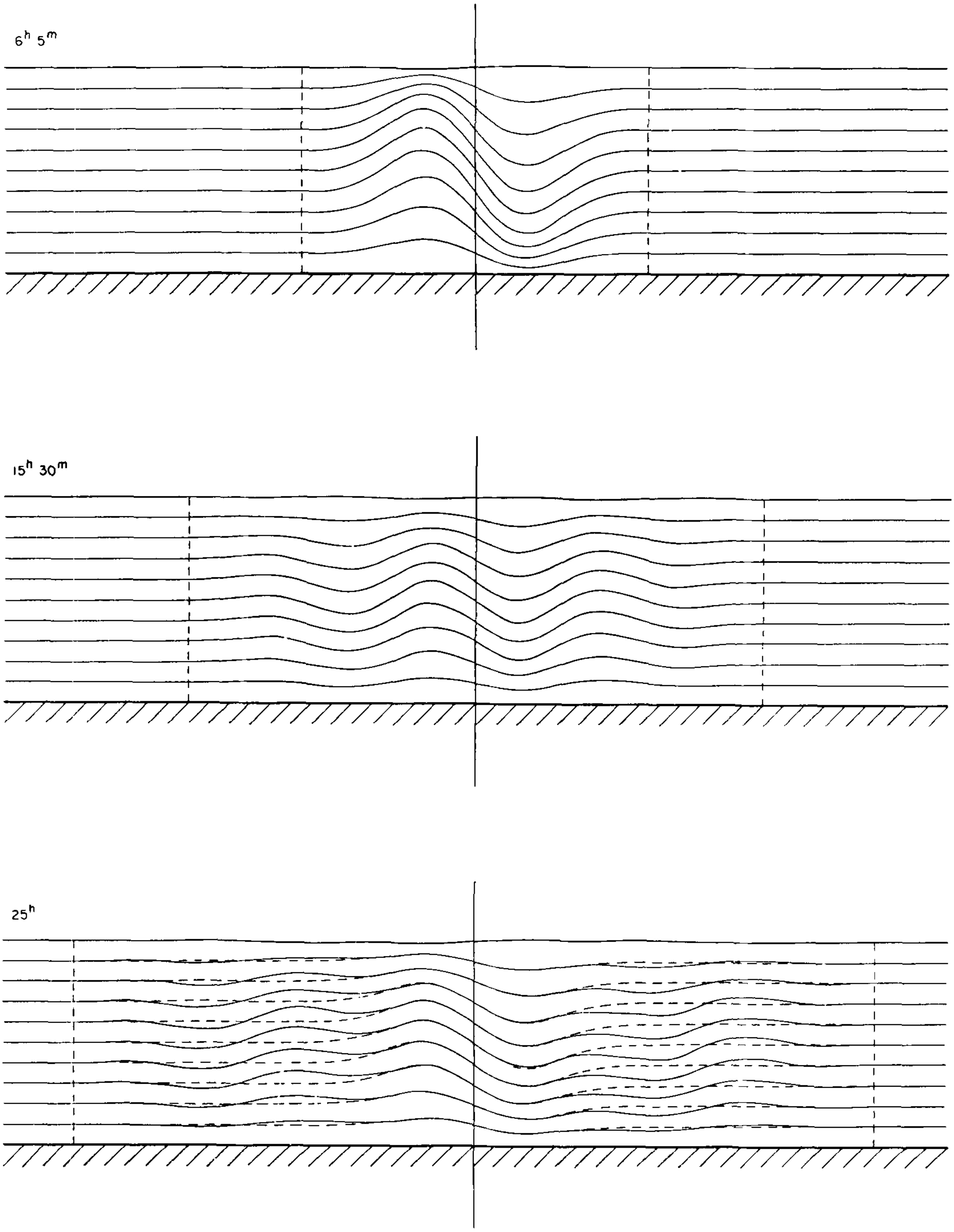


Fig 6. The process of adjustment of the first mode illustrated by three successive cross-sections through the current 6^h5^m, 15^h30^m and 25^h after the instant at which the adjustment started. The solid lines represent the density surfaces at these times and the dashed lines in the third figure are the corresponding density surfaces in the final state of equilibrium. Initially the depth and the width of the current are the same. The dashed vertical lines define the influence region. $\alpha = 0.1$.

these results may have for the numerical forecasting procedure as developed at present (CHARNEY, PHILLIPS, 1953). It seems, however, to the author that the geostrophic approximation may be sufficient when forecasting the mean motion of the atmosphere since the adjustment to geostrophic equilibrium, that is assumed to take place instantaneously, probably means a very slight change of the velocity field. This assumption is less valid when we consider the variation of the velocity with height. We have seen that the adjustment to geostrophic equilibrium of the higher modes by no means takes place instantaneuosly and that the non-geostrophic components, that develop during the adjustment, may be of considerable magnitude. It may then turn out that the assumption of an instantaneous and complete adjustment of the mass field to the new velocity field is one of the major defects of all multiple-parameter models for numerical forecasting purposes.

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